A Magnus theorem for some one-relator groups

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Abstract

We will say that a group G possesses the Magnus property if for any two elements $u, v \in G$ with the same normal closure, u is conjugate to v or v^{-1}. We prove that some one-relator groups, including the fundamental groups of closed nonorientable surfaces of genus $g > 3$ possess this property. The analogous result for orientable surfaces of any finite genus was obtained in [B].

1 Introduction

In 1930 W. Magnus published a very important (for combinatorial group theory and logic) article where he proved the so-called *Freiheitssatz* and the following

Theorem 1.1. [M]. *Let F be a free group and $r, s \in F$. If the normal closures of r and s coincide, then r is conjugate to s or s^{-1}.*

In [BKZ], O. Bogopolski, E. Kudryavtseva and H. Zieschang proved the analogous result for fundamental groups of closed orientable surfaces in case where r and s are represented by simple curves. The suggested proof was geometrical and used coverings, intersection number of curves and Brouwer's fixed-point theorem. However, we were not able to generalize it for arbitrary elements r, s.

Later O. Bogopolski, using algebraic methods in the spirit of Magnus, proved the desired result without restrictions on r, s.

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Theorem 1.2. [B]. *Let G be the fundamental group of a closed orientable surface and $r, s \in G$. If the normal closures of r and s coincide, then r is conjugate to s or s^{-1}.*

In [H], Howie proposed another, topological, proof of this theorem. Both proofs do not work in the nonorientable case. The main result of the present article is the following theorem.

Main Theorem. *Let $G = \langle a, b, y_1, \dots, y_e \,|\, [a, b]uv\rangle$, where $e \geqslant 2$, u, v are non-trivial reduced words in letters $y_1, \dots, y_e$, and u, v have no common letters. Let $r, s \in G$ be two elements with the same normal closures. Then r is conjugate to s or s^{-1}.*

It is known that the fundamental group of a closed nonorientable surface of genus $k \geqslant 3$ has the presentation $\langle x_1, x_2, \dots, x_k \,|\, [x_1, x_2]x_3^2 \cdot \ldots \cdot x_k^2\rangle$. So, we have the following corollary.

Corollary 1.3. *Let G be the fundamental group of a closed nonorientable surface of genus at least 4, and $r, s \in G$. If the normal closures of r and s coincide, then r is conjugate to s or s^{-1}.*

Note that this Corollary trivially holds for genus 1 and 2, but we do not know, whether it holds for genus 3.

We will say that a group G possesses *the Magnus property*, if for any two elements r, s of G with the same normal closures we have that r is conjugate to s or s^{-1}. So, all the above theorems imply that the fundamental group of any compact surface, except of the nonorientable surface of genus 3, possesses the Magnus property.

It was shown in [B], that the Magnus property does not hold for many one-relator groups, including generalized Baumslag-Solitar groups, all noncyclic one-relator groups with torsion, and infinitely many one-relator torsion-free hyperbolic groups.

Now we discuss some logical aspects concerning this property. It was noticed in [B], that if two groups G_1, G_2 are elementary equivalent and G_1 possesses the Magnus property, then G_2 possesses this property too. In particular, any group, which is elementary equivalent to a free group or a free abelian group possesses the Magnus property. This gives another way of proving of Theorem 1.2 and Corollary 1.3. However, there are groups, which are even not existentially equivalent to a free group (hence, they are not limit groups), but possess the Magnus property. The easiest example is the direct product $F_n \times F_m$ of free groups of ranks n, m, where $n + m \geqslant 3$. The other example is the following:

$$G = \langle a, b, x_1, \dots, x_n, y_1, \dots, y_m \,|\, [a, b][X, Y]Z^k\rangle,$$

where $k \geqslant 4$, X, Y are words in the letters $x_1, \dots, x_n$, and Z is a word in the letters $y_1, \dots, y_m$, such that $[X, Y] \neq 1$ and $Z \neq 1$ in the corresponding free groups. This group possesses the Magnus property by our Main Theorem, but is not existentially equivalent to a free group. Indeed, by [CCE], for any $l > 1$ the l-th power of a non-trivial element of a free group can not be expressed as a product of less than $\frac{l+1}{2}$ commutators. Thus, the following formula is valid in G, but is not valid in any free group:

$$\exists\, z_1, z_2, z_3, z_4, z\, (z \neq 1 \wedge [z_1, z_2][z_3, z_4]z^k = 1).$$

Problems. 1) Does every amalgamated product $A *_{\mathbb{Z}} B$, where A, B are free groups and $\mathbb{Z}$ is a maximal cyclic subgroup in both factors, possesses the Magnus property?

2) Does every limit group possesses the Magnus property?

3) Does the group $G = \langle a, b, c \,|\, a^2b^2c^2 \rangle$ possesses the Magnus property?

4) Let A and B be torsion free groups which possess the Magnus property. Does the group $A * B$ possesses the Magnus property? (A positive answer in a partial case can be found in [E]. Note also, that the Magnus property is closed under direct products.)

Some other problems related to the Magnus property are collected in [B].

The plan of this paper is the following. In section 2 we deduce the Main Theorem from Proposition 2.1 and prove auxiliary Lemma 2.2. In section 3 we introduce some technical notions like the left and the right bases of a subgroup, the width of an element, a piece of an element, a special element, and prove auxiliary Lemma 3.2, and Corollary 3.4. In section 4 we present some quotients as amalgamated products and prove the crucial Lemma 4.1. In section 5 we prove Proposition 2.1.

2 Some reduction

First we introduce notations. Let A be a group, $g, h \in A$. The normal closure of g in A is denoted by $\langle\langle g \rangle\rangle_A$ or simply $\langle\langle g \rangle\rangle$ if the group is clear from the context. Denote $[g, h] = g^{-1}h^{-1}gh$. Let X be an alphabet, $x \in X$ and r be a word in the alphabet $X \cup X^{-1}$. By r_x we denote the exponent sum of x in r.

We will deduce the Main Theorem from the following Proposition.

Proposition 2.1. *Let $H = \langle x, b, y_1, \dots, y_e \,|\, [x^k, b]uv \rangle$, where $e \geqslant 2$, $k \neq 0$, u, v are non-trivial reduced words in $y_1, \dots, y_e$, and u, v have no common letters. Let $r, s \in H$ be two elements with the same normal closures and let $r_x = 0$. Then r is conjugate to s or s^{-1}.*

Proof of the Main Theorem. Let $r, s \in G$ and the normal closures of r and s coincide. Suppose that $r_b = 0$. In this case we will use another presentation of G:

$$G = \langle a, b, y_1, \dots, y_e \,|\, [b, a]v^{-1}u^{-1} \rangle.$$

Then the Main Theorem follows immediately from Proposition 2.1.

Now suppose that $r_b \neq 0$. In this case we can embed naturally the group G into the group

$$H = G \underset{a = x^{r_b}}{*} \langle x \,|\, \rangle,$$

where x is a new letter. Clearly, the normal closures of r and s in H coincide.

Claim. The elements r and s are conjugate in H if and only if they are conjugate in G.

Proof. Suppose that $r = h^{-1}sh$, where $h \in H$. Write $h = g_1z_1 \dots g_nz_ng_{n+1}$, where $z_i \in \{x, x^2, \dots, x^{|r_b|-1}\}$, $g_i \in G$ and $g_2, \dots, g_n$ are non-trivial (g_1 and g_{n+1} may be trivial).

We may assume that n is minimal possible. Suppose that $n \geqslant 1$. From the normal form we deduce that $g_1^{-1} s g_1 \in \langle a \rangle$. Then z_1 centralizes this element, that contradicts to the minimality of n. Hence, $n = 0$ and $h \in G$. □

So, we will work now with the group H. This group has the following presentation:

$$\langle x, b, y_1, \dots, y_e \,|\, [x^{r_b}, b]uv \rangle.$$

Let $\bar{b} = x^{r_a} b$. Using Tietze transformation we can rewrite this presentation as

$$\langle x, \bar{b}, y_1, \dots, y_e \,|\, [x^{r_b}, \bar{b}]uv \rangle.$$

Writing r in the generators of this presentation, we have $r_x = r_a r_b - r_b r_a = 0$. Again, by Proposition 2.1, r is conjugate to $s^{\pm 1}$ in H and, hence in G. □

Let $c_1, \dots, c_p$ be the letters of the word u, and $d_1, \dots, d_q$ be the letters of the word v. Consider the following automorphism of H:

$$\psi : \begin{cases} x \mapsto x & \\ b \mapsto bu & \\ c_i \mapsto x^{-k} c_i x^k & (i = 1, \dots, p) \\ d_j \mapsto x^{-k} u^{-1} x^k d_j x^{-k} u x^k & (j = 1, \dots, q). \end{cases}$$

Lemma 2.2. *Let r be a non-trivial element of H such that $r_a = 0$. Then there exists a natural n_0 such that for all $n > n_0$ the element $\psi^n(r)$ is not conjugate to a power of b.*

Proof. Suppose that there exists an m such that $\psi^m(r)$ is conjugate to a power of b. Then for any $l > 0$ the element $\psi^{m+l}(r)$ is conjugate to a power of $b \prod_{i=0}^{l-1} x^{-ik} u x^{ik}$. But the last element is not conjugate to a power of b since its image in $H/\langle\langle b, x \rangle\rangle$ is nontrivial. □

3 Left and right bases

Without loss of generality, we may assume that $k > 0$. Consider the homomorphism $H \to \mathbb{Z}$, which sends x to 1 and any other generator of H to 0. Denote its kernel by N. Denote $g_i = x^{-i} g x^i$ and $Y_i = \{b_i, (y_1)_i, \dots, (y_e)_i\}$. Using the Reidemeister-Schreier method we can find the following presentation of N:

$$N = \langle \cup_{i \in \mathbb{Z}} Y_i \,|\, b_i u_i v_i = b_{i+k} \, (i \in \mathbb{Z}) \rangle.$$

Denote by w_i the word $b_i u_i v_i$. Thus, $w_i = b_{i+k}$ in N. We will use the following three presentations of N depending on a situation:

1. N is the free product of the free groups $G_i = \langle Y_i \,|\, \rangle$ $(i \in \mathbb{Z})$ with amalgamation, where G_i and G_{i+k} are amalgamated over the cyclic subgroup generated by w_i in G_i and by b_{i+k} in G_{i+k}. Denote this cyclic subgroup by Z_{i+k}.

2. $N = N_1 * \dots * N_k$, where $N_l = \dots * G_{l-k} *_{Z_l} G_l *_{Z_{l+k}} G_{l+k} * \dots$, $(l = 1, \dots, k)$. Note that the automorphism ψ leaves each N_l invariant.

3. N is the free group with the free basis $\bigcup_{i\in\mathbb{Z}}(Y_i\setminus\{b_i\})\cup\{b_1,b_2,\dots,b_k\}$. This can be proved with the help of Tietze transformations.

For each $i\leqslant j$ denote $G_{i,j}=\langle G_i,G_{i+1},\dots,G_j\rangle$. The group $G_{i,j}$ has two special free bases

$$\{b_i,b_{i+1},\dots,b_{\min\{i+k-1,j\}}\}\cup\bigcup_{i\leqslant l\leqslant j}(Y_l\setminus\{b_l\})$$

and

$$\bigcup_{i\leqslant l\leqslant j}(Y_l\setminus\{b_l\})\cup\{b_j,b_{j-1},\dots,b_{\max\{j-k+1,i\}}\}$$

which will be called *the left* and *the right basis* of $G_{i,j}$ respectively. The idea behind the left basis is the following: if $i+k\leqslant l\leqslant j$, then we can replace each letter b_l by the word $b_{l-k}u_{l-k}v_{l-k}$. Thus we can eliminate b_l. The idea behind the right basis is analogous: if $i\leqslant l\leqslant j-k$, then we can replace each letter b_l by the word $b_{l+k}v_l^{-1}u_l^{-1}$. In that case we can also eliminate b_l.

Let g be a nontrivial element of N. Consider all the subgroups $G_{i,j}$ such that $g\in G_{i,j}$ and $j-i$ is minimal. There can be several such subgroups (for example if $g=b_kb_{k+1}$, then $g\in G_{k,k+1}$ and $g\in G_{0,1}$). Among these subgroups we choose a subgroup with minimal i. Set $\alpha(g)=i$, and $\omega(g)=j$. The number $||g||=\omega(g)-\alpha(g)+1$ will be called *the width* of the element g.

Definition 3.1. Let $g=z_1z_2\dots z_l$ be the normal form with respect to the decomposition $N=N_1*\dots*N_k$, that is each z_i belongs to some factor of this decomposition and z_i,z_{i+1} do not belong to the same factor. We will call any such z_i *a piece of* g.

We will call g a *special element* of N if

1) g has minimal length l among all its conjugates in N (this means, that z_1 and z_l lie in different factors of this decomposition if $l>1$),

2) if $l=1$, then g has minimal width among all its conjugates in N,

3) no one z_i is conjugate to b.

Note, that if g is a special element and $l>1$, then g written in the right basis of $G_{\alpha(g),\omega(g)}$, is cyclically reduced. Moreover, g has minimal width among all its conjugates in N.

The aim of this section is to prove Corollary 3.4. This will be done with the help of the following Lemma.

Lemma 3.2. *Let g be a special element of N. Then any cyclically reduced word representing the conjugacy class of g in the right (in the left) basis of $G_{\alpha(g),\omega(g)}$ contains a letter from $Y_{\alpha(g)}\setminus\{b_{\alpha(g)}\}$ (from $Y_{\omega(g)}\setminus\{b_{\omega(g)}\}$).*

Proof. We will prove the claim on the right basis of $G_{\alpha(g),\omega(g)}$ only. Let g_R be a cyclically reduced word representing the conjugacy class of g in the right basis of $G_{\alpha(g),\omega(g)}$.

Case 1. Suppose that $||g||\geqslant k$.

Then the right basis of $G_{\alpha(g),\omega(g)}$ does not contain the letter $b_{\alpha(g)}$. Suppose that g_R does not contain a letter from $Y_{\alpha(g)} \setminus \{b_{\alpha(g)}\}$. Then $g_R \in G_{\alpha(g)+1,\omega(g)}$, a contradiction with the minimality of width of g.

Case 2. Suppose that $||g|| < k$.

Then $G_{\alpha(g),\omega(g)} = G_{\alpha(g)} * \dots * G_{\omega(g)} \leqslant N_{\overline{\alpha(g)}} * \dots * N_{\overline{\omega(g)}}$, where $\overline{i}$ denotes the residue of i modulo k. Note that in this case the right basis of $G_{\alpha(g),\omega(g)}$ coincides with $Y_{\alpha(g)} \cup \dots \cup Y_{\omega(g)}$. Consider the pieces of g_R from $N_{\overline{\alpha(g)}}$. Not each of them is a power of $b_{\alpha(g)}$. Therefore g_R contains a letter from $Y_{\alpha(g)} \setminus \{b_{\alpha(g)}\}$. $\square$

Let B be a group, $A \leqslant B$, $C \lhd B$. We will write $A \hookrightarrow B/C$ only in the case when $A \cap C = 1$, meaning the natural embedding. The following theorem is a reformulation of the Magnus Freiheitssatz.

Theorem 3.3. [M]. *Let F be a free group with a basis X, and g be a cyclically reduced word in F with respect to X, containing a letter $x \in X$. Then the subgroup generated by $X \setminus \{x\}$ is naturally embedded into the group $F/\langle\langle g\rangle\rangle$.*

Corollary 3.4. *Let g be a special element of N and j', j be integer numbers such that $j' \leqslant \alpha(g)$ and $\omega(g) \leqslant j$. Then $G_{\alpha(g)+1,j} \hookrightarrow G_{j',j}/\langle\langle g\rangle\rangle$ and $G_{j',\omega(g)-1} \hookrightarrow G_{j',j}/\langle\langle g\rangle\rangle$.*

Proof. We will prove only the first embedding. By Lemma 3.2, the cyclic reduction of g in the right basis of $G_{\alpha(g),\omega(g)}$ contains a letter from $Y_{\alpha(g)} \setminus \{b_{\alpha(g)}\}$. On the other hand, any element of $G_{\alpha(g)+1,j}$ written in this basis does not contain this letter. By Theorem 3.3, $G_{\alpha(g)+1,j} \hookrightarrow G_{\alpha(g),\omega(g)}/\langle\langle g\rangle\rangle$. Now the corollary follows from the fact that $G_{\alpha(g),\omega(g)}$ is a free factor of $G_{j',j}$. $\square$

4 The structure of some quotiens of $G_{n,m}$

Let r be any special element of N. We denote $r_i = x^{-i} r x^i$ for $i \in \mathbb{Z}$. Clearly r_i is a special element. Moreover, $\alpha(r_{i+1}) = \alpha(r_i) + 1$, $\omega(r_{i+1}) = \omega(r_i) + 1$. In particular, all r_i have the same width. Let $j \leqslant i$. Our aim is to present the group $G_{\alpha(r_j),\omega(r_i)}/\langle\langle r_j, r_{j+1}, \dots, r_i\rangle\rangle$ as an amalgamated product. This will be done with the help of Lemma 4.1.

First we introduce a technical notion: *the left and the right sets of words with respect to r_i*. The left set, denoted $L(r_i)$, is $\{w_{\omega(r_i)-k}, \dots, w_{\alpha(r_i)-1}\}$. The right set, denoted $R(r_i)$, is $\{b_{\omega(r_i)}, \dots, b_{\alpha(r_i)-1+k}\}$. We will assume that the subscripts of the elements of these sets are increasing when reading from the left to the right, so these sets are empty if $\omega(r_i) - \alpha(r_i) > k - 1$. Clearly, $L(r_i) \subset G_{-\infty,\alpha(r_i)-1}$ and $R(r_i) \subset G_{\omega(r_i),+\infty}$.

Lemma 4.1. *Let r be a special element of N. Let n, m and i, j be integer numbers such that $j \leqslant i$ and $m \leqslant \alpha(r_j)$, and $\omega(r_i) \leqslant n$. Denote $s = \alpha(r_i)$ and $t = \omega(r_i) - 1$. If $s > t$, we set $G_{s,t} = 1$. Then the following formula holds:*

$$G_{m,n}/\langle\langle r_j, \dots, r_i\rangle\rangle \cong G_{m,t}/\langle\langle r_j, \dots, r_{i-1}\rangle\rangle \underset{\substack{G_{s,t} \\ w_l = b_{l+k}\ (l \in L_{i,m,n})}}{*} G_{s,n}/\langle\langle r_i\rangle\rangle, \qquad (1)$$

where $L_{i,m,n} = \{l \mid w_l \in L(r_i),\, m \leqslant l \leqslant n-k\}$*. Moreover, we have*

$$G_{s+1,n} \hookrightarrow G_{m,n}/\langle\langle r_j, \dots, r_i \rangle\rangle. \tag{2}$$

Proof. Note that (1) implies (2). Indeed, by Corollary 3.4 we have the embedding $G_{s+1,n} \hookrightarrow G_{s,n}/\langle\langle r_i \rangle\rangle$. By (1) we have the embedding $G_{s,n}/\langle\langle r_i \rangle\rangle \hookrightarrow G_{m,n}/\langle\langle r_j, \dots, r_i \rangle\rangle$. Composing these two embeddings, we get the embedding (2).

Now we will prove (1) using induction by $i-j$. For $i-j=0$ the formula (1) has the form

$$G_{m,n}/\langle\langle r_i \rangle\rangle \cong G_{m,t} \underset{\substack{G_{s,t} \\ w_l = b_{l+k}\ (l \in L_{i,m,n})}}{*} G_{s,n}/\langle\langle r_i \rangle\rangle,$$

Let M be the subgroup of $G_{m,n}$ generated by $G_{s,t}$ and the set $\{b_{l+k} \mid l \in L_{i,m,n}\}$. Clearly, M is a subgroup of $G_{m,t}$ and $G_{s,n}$. It is sufficient to prove that M embeds into $G_{s,n}/\langle\langle r_i \rangle\rangle$. Consider two cases.

Case 1. Suppose that $k \geqslant ||r_i||$.

By definition, the group M lies in the subgroup generated by the set $Y_{\alpha(r_i)} \cup \dots \cup Y_{\omega(r_i)-1} \cup \{b_{\omega(r_i)}, \dots, b_{\min\{\alpha(r_i)-1+k,n\}}\}$. In the considered case this set is a part of the left basis of $G_{s,n}$. Consider the cyclically reduced word in this basis, corresponding to r_i. By Lemma 3.1 it contains a letter from $Y_{\omega(r_i)} \setminus \{b_{\omega(r_i)}\}$. Hence, by Theorem 3.3, M embeds into $G_{s,n}/\langle\langle r_i \rangle\rangle$.

Case 2. Suppose that $k < ||r_i||$.

In that case $M = G_{s,t}$ and the desired embedding follows from Corollary 3.4.

Thus, the base of induction holds. Suppose that the formula (1) holds for j, i and prove it for $j, i+1$. Thus, we need to prove that

$$G_{m,n}/\langle\langle r_j, \dots, r_{i+1} \rangle\rangle \cong G_{m,t+1}/\langle\langle r_j, \dots, r_i \rangle\rangle \underset{\substack{G_{s+1,t+1} \\ w_l = b_{l+k}\ (l \in L_{i+1,m,n})}}{*} G_{s+1,n}/\langle\langle r_{i+1} \rangle\rangle. \tag{3}$$

Let M be the subgroup of $G_{m,n}$ generated by $G_{s+1,t+1}$ and the set $\{w_l \mid l \in L_{i+1,m,n}\}$. We can say equivalently that M is generated by $G_{s+1,t+1}$ and the set $\{b_{l+k} \mid l \in L_{i+1,m,n}\}$. It is sufficient to prove that M embeds naturally into the factors of (3), that is into $G_{m,t+1}/\langle\langle r_j, \dots, r_i \rangle\rangle$ and $G_{s+1,n}/\langle\langle r_{i+1} \rangle\rangle$.

The group M can be considered as a subgroup of $G_{s+1,n}$, and $G_{s+1,n}$ naturally embeds into $G_{m,n}/\langle\langle r_j, \dots, r_i \rangle\rangle$ by (2). Hence M naturally embeds into $G_{m,n}/\langle\langle r_j, \dots, r_i \rangle\rangle$. Since $M \leqslant G_{m,t+1} \leqslant G_{m,n}$, it follows that M naturally embeds into $G_{m,t+1}/\langle\langle r_j, \dots, r_i \rangle\rangle$.

The embedding of M into $G_{s+1,n}/\langle\langle r_{i+1} \rangle\rangle$ can be proved by the same argument as in the case of the base of induction. □

The following Lemma can be proved similarly.

Lemma 4.2. *Let r be a special element of N. Let n, m and i, j be integer numbers such that $j \leqslant i$ and $m \leqslant \alpha(r_j)$, and $\omega(r_i) \leqslant n$. Denote $s = \alpha(r_j) + 1$ and $t = \omega(r_i)$. If $s > t$, we set $G_{s,t} = 1$. Then the following formula holds:*

$$G_{m,n}/\langle\langle r_j, \dots, r_i\rangle\rangle \;\cong\; G_{m,t}/\langle\langle r_j\rangle\rangle \underset{\substack{G_{s,t} \\ w_l = b_{l+k}\,(l \in L_{j,m,n})}}{*} G_{s,n}/\langle\langle r_{j+1}, \dots, r_i\rangle\rangle, \tag{4}$$

where $L_{i,m,n} = \{l \mid w_l \in L(r_i),\ m \leqslant l \leqslant n - k\}$.

5 Proof of Proposition 2.1

Let r and s be two elements of H with the same normal closure and $r_x = 0$. Recall that N denotes the kernel of the homomorphism $H \to \mathbb{Z}$, sending x to 1 and each other generator of H to 0. Denote $r_i = x^{-i} r x^i$, $s_i = x^{-i} s x^i$, $i \in \mathbb{Z}$. Then $r_i, s_i \in N$. Moreover, the sets $\mathcal{R} = \{\dots, r_{-1}, r_0, r_1, \dots\}$ and $\mathcal{S} = \{\dots, s_{-2}, s_0, s_1, \dots\}$ have the same normal closure in N. We will prove that some r_i is conjugate to $s_0^{\pm 1}$ in N. This will imply, that r is conjugate to $s^{\pm 1}$ in H.

We may assume, that r and s are special elements. Indeed, let $r = z_1 z_2 \dots z_l$ and $s = c_1 c_2 \dots c_{l'}$ be normal forms with respect to the decomposition $N_1 * \cdots * N_k$. Conjugating, we may assume that the condition 1) of Definition 3.1 is satisfied. Applying a power of an automorphism ψ from Lemma 2.2, we may assume that the condition 3) is satisfied. Finally, if $l = 1$ or $l' = 1$, we may conjugate r or s to ensure the condition 2).

It follows that r_i and s_i are special elements and $\alpha(r_{i+1}) = \alpha(r_i) + 1$, $\omega(r_{i+1}) = \omega(r_i) + 1$. In particular, all r_i have the same width. The same is valid for s_i.

Since s_0 can be deduced from $\mathcal{R}$ in N, there exist integer numbers j, i such that $j \leqslant i$ and s_0 is trivial in $G_{\alpha,\omega}/\langle\langle r_j, r_{j+1}, \dots, r_i\rangle\rangle$, where $\alpha = \alpha(r_j)$, $\omega = \omega(r_i)$. We assume that $i - j$ is minimal possible. By Lemma 4.1 we have

$$G_{\alpha,\omega}/\langle\langle r_j, \dots, r_i\rangle\rangle \;\cong\; G_{\alpha,\omega-1}/\langle\langle r_j, \dots, r_{i-1}\rangle\rangle \underset{A}{*} G_{\alpha(r_i),\omega}/\langle\langle r_i\rangle\rangle,$$

for some subgroup A.

It follows that $s_0 \notin G_{\alpha,\omega-1}$, otherwise s were trivial in $G_{\alpha,\omega-1}/\langle\langle r_j, \dots, r_{i-1}\rangle\rangle$, that contradicts to the minimality of $i - j$. Hence, s_0 written as a word in the left basis of $G_{\alpha,\omega}$ must contain a letter from Y_ω.

Now we will prove that $\alpha(s_0) = \alpha$ and $\omega(s_0) = \omega$. If s_0 contains a letter from $Y_\omega \setminus \{b_\omega\}$, then, clearly, $\omega(s_0) \geqslant \omega$.

Suppose that s_0 contains the letter b_ω, but does not contain any letter from $Y_\omega \setminus \{b_\omega\}$. Then b_ω belongs to the left basis of $G_{\alpha,\omega}$, what can happens only if $\omega - \alpha + 1 \leqslant k$. But in this case s_0 contains a piece, which is a power of b_ω – a contradiction.

Thus, we have proved that $\omega(s_0) \geqslant \omega$. Analogously, $\alpha(s_0) \leqslant \alpha$. Hence, $\alpha(s_0) = \alpha$ and $\omega(s_0) = \omega$. In particular, $||s_0|| = \omega - \alpha + 1 \geqslant ||r_j||$. By symmetry, $||r_j|| \geqslant ||s_0||$. Hence $||r_j|| = ||s_0||$ and $\alpha = \alpha(r_j)$, $\omega = \omega(r_j)$. It follows that s_0 can be deduced from r_j in $G_{\alpha,\omega}$

and the subscript j is determined from the equation $\alpha(s_0) = \alpha(r_j)$. Similarly, r_j can be deduced in $G_{\alpha,\omega}$ from s_0. By Theorem 1.1, s_0 is conjugate to $r_j^{\pm 1}$ in $G_{\alpha,\omega}$. Hence s is conjugate to $r^{\pm 1}$ in H. □

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