Infinite commensurable hyperbolic groups are bi-Lipschitz equivalent

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In [1, p.23] M.Gromov asked the following questions:

1) Under what conditions are two subgroups of finite index in a finitely generated group bi-Lypschitz equivalent? (in particular, is it true that the free groups F_2 and F_3 are bi-Lipschitz equivalent?)

2) Is it true that any two separated nets in hyperbolic space $H^n, n \geq 2$, are bi-Lipschitz equivalent?

It was proved in [2] that free groups of finite rank $n \geq 2$ are bi-Lipschitz equivalent. In the present paper we prove the more general result.

Theorem 1. *Any two infinite commensurable hyperbolic groups are bi-Lipschitz equivalent.*

This is equivalent to

Theorem 1'. *Any two subgroups of finite index in an infinite hyperbolic group are bi-Lipschitz equivalent.*

This theorem answers the first question in the case of hyperbolic groups. The following theorem gives the positive answer to the second question.

Theorem 2. *Any two separated nets* Γ_1 *and* Γ_2 *in hyperbolic space* H^n $(n \geq 2)$ *are bi-Lipschitz equivalent.*

Moreover, there exists a constant $c > 0$ *and a bijection* $\varphi : \Gamma_1 \longrightarrow \Gamma_2$*, such that* $d(x, \varphi(x)) \leq c \ \forall x \in \Gamma_1$ *(here* d *is the standard metric on* H^n*).*

The basic definitions are given in Sec. 1 and Sec. 3; Theorem 1 is proved in Sec. 1 and 2 and Theorem 2 in Sec. 3.

One of the main difficulties in the attempt to answer the second question is that a group may have dead elements with respect to some finite generating system.

Definition 1. An element g of a group G is called *dead* in G with respect to the finite generating set X if $|gx| \leq |g| \ \forall x \in X^{\pm 1}$. Here $|g|$ denotes the length of g in the word metric with respect to X.

Remark. If the language L of all geodesic words of a group G in the alphabet X $(X = X^{-1})$ is regular and M is an automaton recognizing L then the dead words in G with respect to X are precisely those that can be read along the paths leading to the dead states of M (see [3]).

Example 1. It is known that $G = SL_2(Z) = G_1 *_{G_3} G_2$, where $G_1 =< x|x^4 >\cong Z_4$, $G_2 =< y|y^6 >\cong Z_6$ and $G_3 \cong Z_2$. Then y^2 is a dead element of G with respect to $X = \{x, y\}$, since $y^2x = y^{-1}x^{-1}$, $y^2x^{-1} = y^{-1}x$, $y^2y = x^2$. It can be verified analogously that y^{-2} and x^2 are dead elements. There is no other dead element in G with respect to X. Assume $g = g_1g_2 \cdot \ldots \cdot g_n$, where g_i and g_{i+1} lie in the different factors and do not belong to G_3, $1 \leq i \leq n-1$, and the sum $\sum\limits_{i=1}^{n} |g_i|$ is minimal among all such decompositions. It is easy to see that if $n \geq 2$ then $g_i \in \{x^{\pm 1}, y^{\pm 1}\}$. Hence $|g| = n$ and $|gg_{n+1}| = n+1$, where

$g_{n+1} \in \{x^{\pm 1}, y^{\pm 1}\} \setminus \{g_n^{\pm 1}\}$. So, if g is a dead element with respect to X then $n = 1$ and $g \in \{y^2, y^{-2}, x^2\}$.

Example 2. For $G_k = < x, y \,|\, x^3 = y^3 = (xy)^k = 1 >$, $k \geq 3$ it is known that the group G_k is automatic but not hyperbolic for $k = 3$ and hyperbolic for $k \geq 4$. The Caley graph of G_3 with respect to $X = \{x, y\}$ is depicted in Fig. 1. Define w as $(xy)^{k/2}$ if k is even, and as $(xy)^{(k-1)/2}x$ if k is odd. Then the elements w^n, $n \in Z \setminus \{0\}$ are precisely the dead elements of G_k with respect to X. However G_k has no dead elements with respect to the generating set $X = \{x, y, z\}$, where $z = xy$.

In this setting the following questions appear.

Questions. 1) Is it true that for any infinite finitely generated (hyperbolic) group there exists a finite generating set X such that G has no dead words with respect to X ?

2) Let G be a group, X be an arbitrary finite generating system. Is it true that the ratio of the number of dead elements in the ball $B(r) = \{g \in G \,|\, |g| \leq r\}$ to the number of elements in this ball tends to 0 when $r \to \infty$?

3) Let G be an infinite finitely presented group and let X be an arbitrary finite system of generators. Is it true that there exists a constant l (depending on X) such that for any $g \in G$ there exists a $w \in G$ such that $|w| \leq l$ and $|gw| = |g| + 1$?

In view of Lemma 3 the question 3 has an affirmative answer in the class of nonelementary hyperbolic groups.

§1. Preliminary lemmas

Definition 2. Let G_1 and G_2 be finitely generated groups, d_1 and d_2 be word metrics on G_1 and G_2 with respect to some finite generating sets. It is said that G_1 and G_2 are *bi-Lipschitz equivalent*, if there exist a bijection $\varphi : G_1 \longrightarrow G_2$ and a constant β such that

$$\frac{1}{\beta} d_1(g_1, g_2) \leq d_2(\varphi(g_1), \varphi(g_2)) \leq \beta d_1(g_1, g_2) \;\; \forall g_1, g_2 \in G_1.$$

Note that this property doesn't depend on the choice of the finite generating systems.

Definition 3. Groups G_1 and G_2 are *commensurable* if there exists a group which is embeddable into G_1 and G_2 as a subgroup of finite index.

Let G be a group generated by a finite generating set $\mathcal{X}$. Denote by $\Gamma_{\mathcal{X}}(G)$ the right Caley graph of G with respect to $\mathcal{X}$. We consider $\Gamma_{\mathcal{X}}(G)$ as a geodesic metric space whose the metric d is the path metric of $\Gamma_{\mathcal{X}}(G)$ where every edge has the length 1. For any elements a and b from G denote by $[a, b]$ some geodesic path in $\Gamma_{\mathcal{X}}(G)$ with initial point a and terminal point b. Denote by $|a|$ the length of the path $[1, a]$. Let $B_x(r) = \{g \in G \mid d(x, g) \leq r\}$.

A geodesic triangle in $\Gamma_{\mathcal{X}}(G)$ with sides $[a, b]$, $[a, c]$ and $[b, c]$ is called *δ-thin* if for any points $B \in [a, b]$ and $C \in [a, c]$ that satisfy

$$d(a, B) = d(a, C) \leq \frac{1}{2}(d(a, b) + d(a, c) - d(b, c))$$

the inequality $d(B, C) \leq \delta$ holds.

Definition 4. A group G is called *δ-hyperbolic with respect to $\mathcal{X}$* if every geodesic triangle in $\Gamma_{\mathcal{X}}(G)$ is δ-thin.

Let Γ be an infinite elementary hyperbolic group, i.e. Γ has a subgroup of finite index isomorphic to Z. Let us identify this subgroup with Z and let $T = \{t_1, ..., t_m\}$ be a system of right coset representatives of Z in Γ. It's easy to verify that the map $\varphi : \Gamma \longrightarrow Z$, defined by the rule $zt_i \longrightarrow mz + i$, where $t_i \in T, z \in Z$, is a bi-Lipschitz equivalence.

Let Γ be a nonelementary δ-hyperbolic group with respect to the generating set $X = \{\gamma_1, ..., \gamma_n\}$, and Γ_1 be a subgroup of index m. To prove theorem 1', it is sufficient to prove that there exist a constant $c > 0$ and a bijection $\varphi : \Gamma \longrightarrow \Gamma_1$ such that $d(x, \varphi(x)) \leq c$ $\forall x \in \Gamma$.

We will use the fact that a nonelementary hyperbolic group Γ contains a free group of rank 2 [4] and hence there exist the constants $a > 1$ and $r_0 > 0$ such that $\forall r > r_0$,

$$a^r \leq |B(r)| \leq (2n)^{r+1}, \tag{1}$$

where $B(r) = B_1(r)$.

Lemma 1. *For $x, y \in \Gamma$, $x_1 \in [1, x]$, $y_1 \in [1, y]$ and $|x_1| = |y_1|$ the following inequality holds: $d(x_1, y_1) \leq d(x, y) + 2\delta$.*

Proof. If $|x_1| \leq \frac{|x|+|y|-d(x,y)}{2}$, then $d(x_1, y_1) \leq \delta$. Suppose that $|x_1| > \frac{|x|+|y|-d(x,y)}{2}$. Then $d(x_1, x_2) \leq \delta$ and $d(y_1, y_2) \leq \delta$, where x_2 and y_2 are points on a geodesic $[x, y]$ such that $d(x, x_2) = d(x, x_1)$, and $d(y, y_2) = d(y, y_1)$. It follows $d(x_1, y_1) \leq d(x_1, x_2) + d(x_2, y_2) + d(y_2, y_1) \leq d(x, y) + 2\delta$.

Lemma 2. *For an arbitrary $s \geq \delta$ and $x \in \Gamma$ the number of elements $y \in B_x(s)$ such that $|y| \geq |x| + d(x, y) - (4\delta + 2)$ is not less than $|B(s)| - |B(s - \delta)|$.*

Proof. Let $y \in B_x(s)$ and suppose the opposite inequality holds: $|y| < |x| + d(x, y) - (4\delta + 2)$.

Let x_1 and z be points on geodesics $[1, x]$ and $[x, y]$ such that $d(x, x_1) = d(x, z) = 2\delta + 1$. Such points exist and $d(x_1, z) \leq \delta$ since $\frac{1}{2}(|x| + d(x, y) - |y|) > 2\delta + 1$. Hence $d(x_1, y) \leq d(x_1, z) + d(x, y) - d(x, z) \leq d(x, y) - \delta - 1 \leq s - \delta - 1$.

Let x_2 be the vertex on the geodesic $[1, x]$ which is closest to the point x_1 such that $x_1 \in [1, x_2]$. Then $y \in B_{x_2}(s - \delta)$ and the number of such y is no more than $|B(s - \delta)|$.

Definition 5. Let us call a positive number s *beautiful* if for any $x \in \Gamma$ the number of elements $y \in B_x(s)$ such that $|y| \geq |x| + d(x, y) - (4\delta + 2)$ is no less than $|B(\frac{s}{2})|$.

We call the element y an *almost continuation* of the element x.

Lemma 3. *There exists a number k_0 such that for all $k \geq k_0$ at least one beautiful number exists in the interval $[k \log_2 k, k^2]$.*

Proof. If $\delta = 0$, then Γ is a free group [4] and the lemma follows immediately. Assume $\delta > 0$. If the lemma was false, then for all k_0 such that $k_0 \log_2 k_0 \geq \delta$, there would exist a $k \geq k_0$ such that the interval $[k \log_2 k, k^2]$ doesn't contain a beautiful number. Let s be an arbitrary number from this interval. Then $|B(s)| - |B(s - \delta)| < |B(\frac{s}{2})|$. If $s - \delta \in [k \log_2 k, k^2]$, then $|B(s - \delta)| - |B(s - 2\delta)| < |B(\frac{s-\delta}{2})| \leq |B(\frac{s}{2})|$. One can write the series of analogous inequalities the last of which will be $|B(s - t\delta)| - |B(s - (t + 1)\delta)| < |B(\frac{s}{2})|$ where $t \in N$ such that $s - t\delta \geq k \log_2 k > s - (t + 1)\delta$. Adding up these inequalities and taking into account that $|B(s-(t+1)\delta)| \leq |B(k\log_2 k)|$ and $t \leq \frac{k^2}{\delta} - 1$, we deduce that

$$|B(s)| < |B(k\log_2 k)| + \frac{k^2}{\delta}|B(\frac{s}{2})|.$$

For $s = k^2, \frac{k^2}{2}, \ldots, \frac{k^2}{2^p}$, where $p \in N$ is such that $\frac{k^2}{2^p} \geq k\log_2 k > \frac{k^2}{2^{p+1}}$, we obtain the following serie of inequalities:

$$|B(k^2)| < |B(k\log_2 k)| + \frac{k^2}{\delta}|B(\frac{k^2}{2})|,$$

$$|B(\frac{k^2}{2})| < |B(k\log_2 k)| + \frac{k^2}{\delta}|B(\frac{k^2}{4})|,$$

$$\cdots\cdots\cdots$$

$$|B(\frac{k^2}{2^p})| < |B(k\log_2 k)| + \frac{k^2}{\delta}|B(\frac{k^2}{2^{p+1}})|.$$

Combining the inequalities, we get

$$|B(k^2)| < |B(k\log_2 k)|\Big(1 + \frac{k^2}{\delta} + (\frac{k^2}{\delta})^2 + \cdots + (\frac{k^2}{\delta})^p\Big) + (\frac{k^2}{\delta})^{p+1}|B(\frac{k^2}{2^{p+1}})|.$$

If $\frac{k_0^2}{\delta} \geq 2$, we apply $|B(\frac{k^2}{2^{p+1}})| \leq |B(k\log_2 k)|$ and get

$$|B(k^2)| \leq |B(k\log_2 k)| \cdot (\frac{k^2}{\delta})^{\log_2(\frac{k}{\log_2 k})+2}.$$

The last inequality contradicts (1) for sufficiently large k_0.

Applying lemma 3 twice, one gets, that there exist beautiful numbers $k \geq k_0$ and $s \in [k\log_2 k, k^2]$, moreover it is possible to choose k such that: $k > 12\delta + 8$,

$$\frac{k^2+m+1}{k/2-3-4\delta} \leq 3k \qquad (2)$$

and

$$\frac{|B(\frac{k\log_2 k}{2})| - |B(4\delta+3+m)|}{|B(m)|} \geq C_1^2 + C_1, \qquad (3)$$

where $C_1 = (k^2+m+2)|B((3\delta+1)k + 20\delta + 2m + 6)|$.

Let $r = s+m+1$ and foliate Γ by spherical stratums

$$T_j = \{x \in \Gamma \mid r\cdot(j-1) \leq |x| < r\cdot j\}, \quad j = 1, 2, \ldots$$

of width $r-1$. Our first aim is to construct a 1-1 mapping $\varphi_j : T_j \longrightarrow T_{j+1} \bigcap \Gamma_1$, such that $d(x, \varphi_j(x))$ is bounded above by a constant which does not depend on j or $x \in T_j$. This will be done with the help of the constructions below.

§2. Constructions

Let's describe briefly our plan. First for an element $x \in T_j$ we define some element y such that $|y| = [r \cdot j]$ and define the element $y' \in [1, y]$ such that $|x| = |y'|$ and

$$d(x, y') \leq (3\delta + 1)k + 6\delta + 2, \tag{4}$$

$$d(x, y) \leq (3\delta + 1)k + 6\delta + 2 + r. \tag{5}$$

We call an element y the *continuation* of the element x.

The *construction*, constructed on the *source* x is the set

$$S_x = O_m(\{z \in B_y(s) \; \Big| \; |z| \geq |y| + d(z, y) - (4\delta + 2)\,; \; z \notin B_y(4\delta + 3 + m)\}) \bigcap \Gamma_1 \tag{6}$$

Here O_m denotes the m-neighbourhood of the set in the brackets. It is clear that $S_x \subseteq T_{j+1}$.

It will be proved later that

1) the number of sources producing the same construction is bounded above by some constant $C = C(k, \delta, n, m)$,

2) in every construction C elements can be choosen such that the elements choosen in different constructions are pairwise different.

This will permit us to construct a 1-1 correspondence $\varphi_j : T_j \longrightarrow T_{j+1} \bigcap \Gamma_1$, associating with every source $x \in T_j$ some element from its construction S_x. In addition,

$$d(x, \varphi_j(x)) \leq d(x, y) + d(y, \varphi_j(x)) \leq (3\delta + 1)k + 6\delta + 2s + m + 3. \tag{7}$$

So, be $x \in T_j$. Since k is a beautiful number (compare with definition 5) there exists a sequence of the elements $x = x_0, x_1, \ldots$ from Γ such that

$$\frac{k}{2} - 1 \leq d(x_i, x_{i+1}) \leq k\,; \; |x_{i+1}| \geq |x_i| + d(x_i, x_{i+1}) - (4\delta + 2), \tag{8}$$

$i = 1, 2, ...$ (see Fig. 2 further).

Let t be a number such that $x_t \in T_j, x_{t+1} \in T_{j+1}$, then

$$r \geq |x_t| - |x_0| = \sum_{i=0}^{t-1} (|x_{i+1}| - |x_i|) \geq t \cdot (\frac{k}{2} - 3 - 4\delta)$$

and hence $t \leq 3k$ because of (2).

Let x'_i be a point on the geodesic $[1, x_i]$, such that $|x'_i| = |x|$, $1 \leq i \leq t$. It exists since $|x_i| \geq |x_{i-1}| + \frac{k}{2} - 3 - 4\delta \geq |x_{i-1}| \geq \cdots \geq |x|$.

Let y be the first vertex on the geodesic $[x_t, x_{t+1}]$ such that $|y| = [r \cdot j]$. Let y' be the vertex on the geodesic $[1, y]$ such that $|y'| = |x|$. Below we estimate the distances $d(x, x'_1)$, $d(x'_i, x'_{i+1})$ and $d(x'_t, y')$, $1 \leq i \leq t - 1$.

Let e and e_1 be points on the geodesics $[1, x]$ and $[1, x_1]$ such that $|e| = |e_1| = \frac{1}{2}(|x| + |x_1| - d(x, x_1))$. Then $d(e, e_1) \leq \delta$, and in view of (8) we have

$$d(x,x'_1)=d(x,e)+d(e,e_1)+d(e_1,x'_1)\le 2(|x|-|e|)+\delta\le 5\delta+2.$$

If $i\ge 1$ then

$$|x_i|-|x'_i|=\sum_{l=0}^{i-1}(|x_{l+1}|-|x_l|)\ge i\cdot\big(\frac{k}{2}-3-4\delta\big)\ge 2\delta+1.$$

In view of (8), $\frac{1}{2}(|x_i|+|x_{i+1}|-d(x_i,x_{i+1}))\ge|x_i|-2\delta-1\ge|x'_i|=|x'_{i+1}|$. So $d(x'_i,x'_{i+1})\le\delta$. In view of lemma 1 and the inequalities (8), we have $d(x'_t,y')\le d(x_t,y)+2\delta\le d(x_t,x_{t+1})+2\delta\le k+2\delta$, hence $d(x,y')\le d(x,x'_1)+\sum\limits_{i=1}^{t-1}d(x'_i,x'_{i+1})+d(x'_t,y')\le 5\delta+2+(t-1)\delta+k+2\delta\le(3\delta+1)k+6\delta+2$.

Since $d(y',y)\le r$ we get $d(x,y)\le(3\delta+1)k+6\delta+2+r$.

Let's now define a construction S_x by the formula (6). Suppose that two constructions have a common element: $h\in S_x\bigcap S_u$. Let y and v be the continuations of the sources x and u and z_1 and z_2 be elements in the sets

$$\{z\in B_y(s)\;\big|\;|z|\ge|y|+d(z,y)-(4\delta+2)\,;\;z\notin B_y(4\delta+3+m)\}$$

and

$$\{z\in B_v(s)\;\big|\;|z|\ge|v|+d(z,v)-(4\delta+2)\,;\;z\notin B_v(4\delta+3+m)\},$$

such that $d(h,z_1)\le m$ and $d(h,z_2)\le m$. Let z'_1 and z'_2 be the points on the geodesics $[1,z_1]$ and $[1,z_2]$ respectively such that $|z'_1|=|z'_2|=[r\cdot j](=|y|=|v|)$. We can prove the inequalities $d(y,z'_1)\le 5\delta+2$ and $d(v,z'_2)\le 5\delta+2$ by the same reasoning as the inequality $d(x,x'_1)\le 5\delta+2$ was proved.

Since $d(z_1,z_2)\le 2m$, then we get $d(z'_1,z'_2)\le 2m+2\delta$ by lemma 1. From this $d(y,v)\le 12\delta+2m+4$.

Let u'' and u' be vertices on the geodesics $[1,y]$ and $[1,v]$, such that $|u''|=|u'|=|u|$. By lemma 1, $d(u'',u')\le d(y,v)+2\delta\le 14\delta+2m+4$.

In view of (4), $d(u,u')\le(3\delta+1)k+6\delta+2$, and so $d(u,u'')\le(3\delta+1)k+20\delta+2m+6$.

Let w be the point on the geodesic $[1,y]$ such that $|w|=r\cdot(j-1)$. Then $u''\in[w,y]$, hence u lies in the $((3\delta+1)k+20\delta+2m+6)$ - neighbourhood of the geodesic $[w,y]$. Hence the number of the sources from which a construction intersecting with a given construction S_x can appear, does not exceed $C=(r+1)\cdot|B((3\delta+1)k+20\delta+2m+6)|$.

In particular, a given construction can not occur from more than C sources and the number of constructions intersecting with a given one also does not exceed C.

Observe that the distance between the sources x and u may be too great: $d(x,u)\le d(x,y')+d(y',u'')+d(u'',u)\le((3\delta+1)k+6\delta+2)+r+((3\delta+1)k+20\delta+2m+6)$, and hence it can not be used in the following calculations; so we use the estimation for $d(u,[w,y])$.

Since s is a beautiful number, the number of elements in the set

$$\{z\in B_y(s)\;\big|\;|z|\ge|y|+d(z,y)-(4\delta+2)\,;\;z\notin B_y(4\delta+3+m)\}$$

isn't less than $|B(\frac{s}{2})| - |B(4\delta + 3 + m)|$. Since in the m-neighbourhood of any element from Γ we can find some element from Γ_1, this set is contained in the m-neighbourhood of the set S_x. So

$$|S_x| \geq \frac{|B(\frac{s}{2})| - |B(4\delta + 3 + m)|}{|B(m)|} \geq \frac{|B(\frac{k \log_2 k}{2})| - |B(4\delta + 3 + m)|}{|B(m)|} \geq C^2 + C \qquad (9)$$

because of the choice of k (see (3)).

Let's prove that in every construction it is possible to choose C elements such that the following condition are fulfilled:

the samplings don't intersect if the constructions are different. (10)

Let's enumerate the constructions lying in the $(j+1)$-th stratum. In the first consruction we can choose C elements because of (9). Assume that we have already chosen C elements in every of the constructions $S_1, \dots, S_i$ such that the condition (10) is satisfied for them. The construction S_{i+1} can intersects with not more than C other constructions and hence it doesn't contain more than the C^2 elements chosen before. In view of (9), in S_{i+1} we can choose C elements such that condition (10) is satisfied.

This allows us to construct a 1-1 correspondence $\varphi_j : T_j \longrightarrow T_{j+1} \bigcap \Gamma_1$ for every j satisfying condition (7). Let's now define the mappings

$$\varphi_{j,1} = \varphi_j \mid (T_j \setminus \Gamma_1) \bigcup Im\varphi_{j-1,1} \ \ , \ \ \varphi_{j,2} = id \mid T_j \bigcap (\Gamma_1 \setminus Im\varphi_{j-1,1}).$$

Finally we define the mapping $\varphi : \Gamma \longrightarrow \Gamma_1$ by the rule $\varphi(x) = \varphi_{j,i}(x)$ if $x \in dom\varphi_{j,i}$. It is easy to verify that φ is a bijection and in view of (7) we get

$$d(x, \varphi(x)) \leq (3\delta + 1)k + 6\delta + 2s + m + 3.$$

Theorem 1' is proved.

§3. Separated nets
in the hyperbolic space H^n

Definition 6. Let (X, d) be a metric space. A subset $Y \subseteq X$ is called a *separated net* in X if there exist a constants $\varepsilon > 0$ and $\mu > 0$ such that the following conditions are satisfied:

1) $\forall x \in X \ \exists y \in Y : d(x, y) \leq \varepsilon$,

2) $\forall y_1, y_2 \in Y$, if $y_1 \neq y_2$, then $d(y_1, y_2) \geq \mu$.

Subgroups of finite index in G can be considered as separated nets in G. So it is not surprising that the proof of the theorem $1'$ can be modified to the proof of the theorem 2.

Proof of the theorem 2. Let H^n denote hyperbolic space of dimension $n \geq 2$, d be its metric and δ its hyperbolicity constant. Let $V(r)$ denotes the volume of the ball of radius r in H^n. It is known that $V(r)$ growth exponentially [5].

If G is an arbitrary μ -separated ε -net in H^n and $r > \varepsilon$ then it follows easy from the definition that

$$\frac{V(r-\varepsilon)}{V(\varepsilon)} \leq |B_x(r) \bigcap G| \leq \frac{V(r+\frac{\mu}{2})}{V(\frac{\mu}{2})}.$$

It follows from this that there exist positive constants a, b and c, such that for any $r \geq 0$ the following inequality is satisfied

$$ae^{(n-1)r} - b \leq |B_x(r) \bigcap G| \leq ce^{(n-1)r}. \tag{11}$$

Let 1 be an arbitrary point in H^n and $|x| = d(1, x)$ for $x \in H^n$. Analogously to lemma 2 we can prove the

Lemma 2′. *For an arbitrary $\alpha \geq 0, s \geq 0$ and $x \in H^n$ the number of elements $y \in B_x(s) \bigcap G$ such that $|y| \geq |x| + d(x, y) - \alpha\delta$ is not less than*

$$|B_x(s) \bigcap G| - ce^{(n-1)(s - \frac{\alpha-2}{2}\delta)}.$$

Choose α such that $\alpha > \frac{2\ln(\frac{c}{a})}{(n-1)\delta} + 2$.

Definition 5'. We call a positive real number s *beautiful for the net G* if $\forall x \in H^n$ the number of elements $y \in B_x(s) \bigcap G$ such that $|y| \geq |x| + d(x, y) - \alpha\delta$, is not less than $|B_x(\frac{s}{2}) \bigcap G|$.

Lemma 3′. *There exists a number s_0, such that every $s \geq s_0$ is a beautiful number for the net G.*

Proof. In view of lemma 2′ it is sufficient to prove that for large s the following inequality holds:

$$|B_x(s) \bigcap G| - ce^{(n-1)(s - \frac{\alpha-2}{2}\delta)} \geq |B_x(\frac{s}{2}) \bigcap G|.$$

This is actually so because of the choice of α and inequality (11).

Now let Γ_1 and Γ_2 be two arbitrary separated nets in H^n with constants $\mu_i, \varepsilon_i, a_i, b_i$ and c_i $(i = 1, 2)$. Choose a beautiful number s for the net Γ_1 such that the following inequality holds:

$$a_1 e^{(n-1)\frac{s}{2}} - b_1 - c_1 e^{(n-1)\alpha\delta} \geq C^2 + C, \tag{12}$$

where

$$C = (\frac{s+1}{4(\alpha+2)\delta} + 2)(c_1 + c_2)e^{4(\alpha+2)\delta(n-1)}.$$

In the case of H^n the definition of the constructions is simple since in H^n any geodesic can be continued infinitely in both directions. Let $r = s + 1$,

$$T_j = \{x \in H^n \;\mid\; r \cdot (j-1) \leq |x| < r \cdot j\}, \quad j = 1, 2, \ldots.$$

For the sources $x \in (\Gamma_1 \bigcup \Gamma_2) \bigcap T_j$ and only for them define the construction $S_x \subseteq T_{j+1}$ in the following way:

$$S_x = \{z \in B_y(s) \bigcap \Gamma_1 \;\Big|\; |z| \geq |y| + d(z,y) - \alpha\delta\,;\; z \notin B_y(\alpha\delta)\},$$

where y is a point such that $x \in [1,y]$ and $|y| = r \cdot j$.

Suppose that two constructions intersect: $z \in S_x \bigcap S_u$. Let v be a point such that $u \in [1,v]$ and $|v| = r \cdot j$ and z' be a point on geodesic $[1,z]$ such that $|z'| = r \cdot j$.

By analogy with the proof of the inequality $d(x, x_1^{'}) \leq 5\delta + 2$ in §2, we can prove the inequalities $d(y,z') \leq (\alpha+1)\delta$ and $d(z',v) \leq (\alpha+1)\delta$. It follows that $d(y,v) \leq 2(\alpha+1)\delta$. Let w be a point on the geodesic $[1,y]$ such that $|w| = r \cdot (j-1)$ and u' be a point on the geodesic $[w,y]$ such that $|u'| = |u|$. Then lemma 1 applied to the triangle $[1,y,v]$ yields $d(u,u') \leq d(y,v) + 2\delta \leq 2(\alpha+2)\delta$. Define $A = 2(\alpha+2)\delta$.

So $u \in (\Gamma_1 \bigcup \Gamma_2) \bigcap T_j$ lies in A-neighbourhood of the geodesic $[w,y]$ and hence it lies in the union of at most $\frac{r}{2A} + 2$ balls of radius $2A$ covering this geodesic of length r. So the number of such sourses u is at most

$$(\frac{r}{2A} + 2)|B(2A) \bigcap (\Gamma_1 \bigcup \Gamma_2)| \leq (\frac{s+1}{2A} + 2)(c_1 + c_2)e^{2A(n-1)} = C.$$

It follows from this that

1) The number of constructions, intersecting the given one is at most C,

2) every construction appears from not more than C sources,

3) $|S_x| \geq |B_y(\frac{s}{2}) \bigcap \Gamma_1| - |B_y(\alpha\delta) \bigcap \Gamma_1| \geq a_1 e^{(n-1)\frac{s}{2}} - b_1 - c_1 e^{(n-1)\alpha\delta} \geq C^2 + C$, since s is a beautiful number and C satisfies (12).

Reasons analogous to those which were developed at the end of §2, permit us to construct a bijection $\varphi_1 : \Gamma_1 \bigcup \Gamma_2 \longrightarrow \Gamma_1$, such that $d(x, \varphi_1(x)) \leq 2r \;\; \forall x \in \Gamma_1 \bigcup \Gamma_2$. Analogously we can find a constant t and a bijection $\varphi_2 : \Gamma_1 \bigcup \Gamma_2 \longrightarrow \Gamma_2$ such that $d(x, \varphi_2(x)) \leq 2t \;\; \forall x \in \Gamma_1 \bigcup \Gamma_2$. Then $\varphi = \varphi_1^{-1} \circ \varphi_2 : \Gamma_1 \longrightarrow \Gamma_2$ is bijection such that $d(x, \varphi(x)) \leq 2(r+t)$.

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