MULTIVARIATE R-O VARYING MEASURES
PART II: INDIVIDUAL BOUNDS

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ABSTRACT. For a multivariate R-O varying measure sharp bounds for a fixed direction of the truncated moment functions and tail moment functions are given. They improve the uniform results in part I of that paper. Furthermore, it is shown that these functions are R-O varying and Karamata like results on the ratio of these functions hold true.

1. INTRODUCTION

In the paper [13] we introduced the notain of R-O varying measures and carefully derived a complete set of uniform bounds for the tail moment functions and the truncated moment functions defined in (1.3) and (1.4) below, respectively. In fact it is shown that these functions are uniformly R-O varying (see [14]) for any compact set of directions and Karamata like results on the asymptotic behavior of the ratio of these functions hold true.

See [13] for further information on regular variation in $\mathbb{R}^d$, its historical development and its applications. Let us give a brief review of the main definitions and results of [13]:

By [3] a Borel measurable function $f : (0,\infty) \to \mathrm{GL}(\mathbb{R}^d)$ is said to vary regularly with index F (a $d \times d$ matrix) if for $\lambda > 0$

$$\lim_{t\to\infty} f(\lambda t) f(t)^{-1} = \lambda^F. \tag{1.1}$$

Here $\lambda^F = \exp(F \log \lambda)$ and $\exp(A) = \sum_{k=0}^{\infty} \frac{1}{k!} A^k$. We write $f \in \mathrm{RV}(F)$ if (1.1) holds. Note that by Theorem 2.2 of [3] the convergence in (1.1) is uniform on compact subsets of $\{\lambda > 0\}$.

A finite positive measure μ on $\mathbb{R}^d$ is called R-O varying with index F, if there exist a function $f \in \mathrm{RV}(-F)$, a Borel measurable function $k : (0,\infty) \to (0,\infty)$ with $k(t+1)/k(t) \to c \geq 1$ as $t \to \infty$ and a σ-finite measure φ on $\Gamma = \mathbb{R}^d \setminus \{0\}$ such that

$$k(t)(f(k(t))\mu) \to \varphi \quad \text{as } t \to \infty. \tag{1.2}$$

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Here, for $A \in \mathrm{GL}(\mathbb{R}^d)$, $(A\mu)$ denotes the image measure defined by $(A\mu)(B) = \mu(A^{-1}B)$ and the convergence in (1.2) is the vage convergence of σ-finite measures on Γ. We write $\mu \in \mathrm{ROV}(F, c)$ if (1.2) holds. If $k(t) = t$ then μ is called regularly varying and we write $\mu \in \mathrm{RVM}(F)$ in this case.

Regularly varying measures or more generally R-O varying measures were introduced to analyze the so called generalized domains of attraction of operator stable and operator semistable laws, respectively. See [5] and [11]. The resulting theory is so powerful that it has found numerous applications in probability theory and might be useful in other areas of mathematics. See e.g [7], [8], [9], [6] and the literature cited in [13].

In the fundamental paper [13], for a measure $\mu \in \mathrm{ROV}(F, c)$ we investigated the asymptotic behavior of the tail moment functions

$$V_a(t, \theta) = \int_{|\langle x,\theta\rangle|>t} |\langle x, \theta\rangle|^a d\mu(x) \tag{1.3}$$

and the truncated moment functions

$$U_b(x, \theta) = \int_{|\langle x,\theta\rangle|\leq t} |\langle x, \theta\rangle|^b d\mu(x) \tag{1.4}$$

for directions θ belonging to a compact subset of Γ as $t \to \infty$. It is shown there that for certain values of a, b both V_a and U_b are uniformly R-O varying (see [14] and Definition 4.1 of [13]) with indices only depending on the smallest and largest real part of the eigenvalues of the index F. Roughly speaking, by Corollary 4.21 of [13] we have for any compact $K \subset \Gamma$ and any $\delta > 0$, if $a < 1/a_p$ and $b > 1/a_1$ that for some positive constants B_1, B_2, C_1, C_2

$$B_1 t^{a-\frac{1}{a_1}-\delta} \leq V_a(t, \theta) \leq B_2 t^{a-\frac{1}{a_p}+\delta}$$

and

$$C_1 t^{b-\frac{1}{a_1}-\delta} \leq U_b(t, \theta) \leq C_2 t^{b-\frac{1}{a_p}+\delta}$$

for all large t, uniformly in $\theta \in K$. Here $0 < a_1 < \cdots < a_p$ denote the real parts of the eigenvalues of F.

However, if the direction θ is fixed these bounds are not very sharp since usually $a_1 < a_p$. The purpose of this paper is to improve the bounds in [13] for a fixed $\theta \in \Gamma$. In fact we will show that there exists an index function $\alpha(\theta)$ with values in $\{a_1^{-1}, \ldots, a_p^{-1}\}$ such that roughly speaking $V_a(t, \theta) \approx t^{a-\alpha(\theta)}$ and $U_b(t, \theta) \approx t^{b-\alpha(\theta)}$ whenever $a < \alpha(\theta) < b$. This strongly refines the uniform bounds obtained in [13].

These sharp bounds are essential for various deep theorems on measures attracted to operator semistable laws, including laws of the iterated logarithms. See [12] and [1].

The proof of our main result (Theorem 3.5) relies heavily on the uniform bounds obtained in [13] together with the spectral decomposition in [10].

In section 2 some notation as well as the basic results on regular variation on $\mathrm{GL}(\mathbb{R}^d)$ including the spectral decomposition are collected. Section 3 contains the main results together with several technical results necessary for the proof.

2. Notation and preliminary results

Let $\mathcal{M} = \mathcal{M}(\Gamma)$ denote the set of all σ-finite nonnegative Borel measures on Γ which are finite outside every neighborhood of the origin. We equip $\mathcal{M}$ with the following topology: For $\mu_n, \mu \in \mathcal{M}$ we say

$$\mu_n \to \mu \quad \text{as } n \to \infty$$

if and only if

$$\mu_n(A) \to \mu(A) \quad \text{as } n \to \infty$$

for all Borel sets $A \subset \Gamma$ bounded away from the origin which are continuity sets of the limit measure μ, e.g. $\mu(\partial A) = 0$ where ∂A denotes the topological boundary of A. Since the measures in $\mathcal{M}$ are finite outside every neighborhood of the origin this is the vage convergence on the space $\overline{\mathbb{R}^d} \setminus \{0\}$ where $\overline{\mathbb{R}^d}$ is the one-point compactification of $\mathbb{R}^d$.

Definition 2.1. (cf. [3]). A measure μ in $\mathcal{M}$ is said to vary regularly with index F if there exists a function $f \in \mathrm{RV}(-F)$ such that

$$t \cdot (f(t)\mu) \to \varphi \quad \text{as } t \to \infty \tag{2.1}$$

for some $\varphi \in \mathcal{M}$ not supported on any proper subspace of $\mathbb{R}^d$. In this case we write $\mu \in \mathrm{RVM}(F)$.

It follows from the regular variation of f that φ satisfies the transformation formula

$$(\lambda^F \varphi) = \lambda \cdot \varphi \quad \text{for } \lambda > 0. \tag{2.2}$$

It is often convenient to work with sequences instead of a continuous parameter in (2.1). These two different approaches are equivalent as outlined below. A sequence $(B_n) \subset \mathrm{GL}(\mathbb{R}^d)$ is called regularly varying with index F if and only if the function $f(t) = B_{[t]}$, where $[t]$ denotes the integer part of t is in $\mathrm{RV}(F)$. It follows from the uniform convergence in (1.1) that this is equivalent to $B_{[\lambda n]} B_n^{-1} \to \lambda^F$ as $n \to \infty$. With a little abuse of notation we write $(B_n) \in \mathrm{RV}(F)$ in this case too.

By Lemma 2.2 of [13] $\mu \in \mathrm{RVM}(F)$ if and only if there exists a sequence $(B_n) \in \mathrm{RV}(-F)$ such that

$$n \cdot (B_n \mu) \to \varphi \quad \text{as } n \to \infty. \tag{2.3}$$

Now we will extend definition 2.1 to R-O varying measures:

Definition 2.2. $\mu \in \mathcal{M}$ is R-O varying if there exists a function $f \in \mathrm{RV}(-F)$ and a Borel measurable function $k : (0, \infty) \to (0, \infty)$ with $k(t) \to \infty$ as $t \to \infty$ and $k(t + 1)/k(t) \to c \geq 1$ as $t \to \infty$ such that

$$k(t) \cdot (f(k(t))\mu) \to \varphi \quad \text{as } t \to \infty, \tag{2.4}$$

for some $\varphi \in \mathcal{M}$ not supported on any proper subspace of $\mathbb{R}^d$. We say that $\mu \in \mathrm{ROV}(F, c)$.

As for regularly varying measures the limit measure in (2.4) exhibits a certain transformation property. In fact (2.4) and that regular variation of f gives that

$$c^F \varphi = c \cdot \varphi. \tag{2.5}$$

As before there is a discrete analogue to Definition 2.2: $\mu \in \mathrm{ROV}(F, c)$ if and only if there exist a sequence $(B_n) \in \mathrm{RV}(-F)$ and a sequence of natural numbers (k_n) tending to infinity with $k_{n+1}/k_n \to c$ such that

$$k_n \cdot (B_{k_n}\mu) \to \varphi \quad \text{as } n \to \infty. \tag{2.6}$$

For further properties of R-O varying measures see section 3 of [13].

Another fact we need in the proof of our main results is the so called *spectral decomposition* for regularly varying functions (resp. sequences) in $\mathrm{GL}(\mathbb{R}^d)$, proved in [10]. It provides a powerful tool that decomposes the underlying vector space and gives information about the growth rate of the regularly varying function along different directions. Since we will only work with sequences, we restate here an equivalent version of the theorems in [10] for regularly varying sequences for sake of completeness.

Assume that $(B_n) \in \mathrm{RV}(F)$. Factor the minimal polynomial of F into $f_1(x) \cdots f_p(x)$ where all roots of f_i have real part a_i and $a_i < a_j$ for $i < j$. Define $V_i = \mathrm{Ker}(f_i(F))$. Then $V_1 \oplus \cdots \oplus V_p$ is a direct sum decomposition of $\mathbb{R}^d$ into F–invariant subspaces, and we may write $F = F_1 \oplus \cdots \oplus F_p$ where $F_i : V_i \to V_i$ and every eigenvalue of F_i has real part equal to a_i. We will call this the *spectral decomposition of* $\mathbb{R}^d$ *relative to* F. This is a special case of the primary decomposition theorem of linear algebra. See for example [2].

The next two results are stated in [10]. We include them here for sake of completeness.

Theorem 2.3. *Suppose* (B_n) *is a regularly varying sequence with index* F *and let* $\mathbb{R}^d = \tilde{V}_1 \oplus \cdots \oplus \tilde{V}_p$ *be the spectral decomposition of* $\mathbb{R}^d$ *relative to* F*. Then there exists a nested sequence of subspaces* $\tilde{L}_1 \subset \cdots \subset \tilde{L}_p = \mathbb{R}^d$ *such that for each* $i = 1, \ldots, p$ *we have*

(a) $\dim \tilde{L}_i = \dim(\tilde{V}_1 \oplus \cdots \oplus \tilde{V}_i)$*;*
(b) *if* $x \in \tilde{L}_i$*, then* $B_n x/\|B_n x\| \to \tilde{V}_1 \oplus \cdots \oplus \tilde{V}_i$*;*
(c) *if* $x \notin \tilde{L}_i$*, then* $B_n x/\|B_n x\| \to \tilde{V}_{i+1} \oplus \cdots \oplus \tilde{V}_p$*;*
(d) *if* $x \in \tilde{L}_i$*, then* $n^{-\rho}\|B_n x\| \to 0$ *for all* $\rho > a_i$*;*

(e) *if* $x \notin \tilde{L}_i$*, then* $n^{-\rho}\|B_n x\| \to \infty$ *for all* $\rho < a_{i+1}$.

The next result is the basic characterization of regular variation for sequences of linear operators. Again it is stated in [10]. Let us agree to write $D_n \sim B_n$ if $D_n B_n^{-1} \to I$.

Theorem 2.4. *(B_n) varies regularly with index F if and only if $B_n \sim D_n T$ for some $T \in \mathrm{GL}(\mathbb{R}^d)$ and some (D_n) regularly varying with index F such that: each $\tilde{V}_i$ in the spectral decomposition of $\mathbb{R}^d$ with respect to F is D_n–invariant; and $D_n = D_n^{(1)} \oplus \cdots \oplus D_n^{(p)}$ where each $D_n^{(i)} : \tilde{V}_i \to \tilde{V}_i$ is regularly varying with index F_i.*

3. Individual bounds

In section 4 of [13] we considered the uniform R-O variation of the truncated moment functions U_b and the tail moment functions V_a over a compact set of directions. Roughly speaking it turned out that the rate at which these functions either grow or decay at infinity is controlled by the largest and smallest real part of the eigenvalues of the index. (See Corollary 4.21 of [13].)

However, if the direction θ is fixed sharper bounds can be obtained. In fact we will show that there exists an index function $\alpha(\theta)$ such that $V_a(t,\theta) \approx t^{a-\alpha(\theta)}$ and $U_b(t,\theta) \approx t^{b-\alpha(\theta)}$ whenever $a < \alpha(\theta) < b$ which refines the results of Corollary 4.21 of [13].

Our approach will utilize the spectral decomposition (Theorem 2.4) along with the uniform bounds of section 4 of [13]. Before we state the main results of this section we first argue how to reduce the problem to a simpler case and proof some technical lemmas.

Let $\mu \in \mathrm{ROV}(F, c)$ for some $c > 1$ and let $\mathbb{R}^d = V_1 \oplus \cdots \oplus V_p$ be the spectral decomposition with respect to F into F-invariant subspaces. Then $F = F_1 \oplus \cdots \oplus F_p$ where $F_i : V_i \to V_i$ and every eigenvalue of F_i has real part a_i, where $0 < a_1 < \cdots < a_p$ denotes the real spectrum of F. By (2.6) there exists a sequence $(B_n) \in \mathrm{RV}(-F)$ and a sequence (k_n) with $k_{n+1}/k_n \to c$ such that $k_n(B_{k_n}\mu) \to \varphi$. Apply Theorem 2.4 to obtain a $T \in \mathrm{GL}(\mathbb{R}^d)$ and a $(D_n) \in \mathrm{RV}(-F)$ such that $B_n \sim D_n T$ where $D_n = D_n^{(1)} \oplus \cdots \oplus D_n^{(p)}$ and each $D_n : V_i \to V_i$ is $\mathrm{RV}(-F_i)$.

Let $\bar{\mu} = T(\mu)$. Then $k_n(D_{k_n}\bar{\mu}) \to \varphi$ to, so $\bar{\mu} \in \mathrm{ROV}(F, c)$ too and the norming operators are block diagonal with respect to the direct sum decomposition of $\mathbb{R}^d$. Let $\bar{V}_a(t,\theta) = \int_{|\langle x,\theta\rangle|>t} |\langle x,\theta\rangle|^a d\bar{\mu}(x)$ and $\bar{U}_b(t,\theta) = \int_{|\langle x,\theta\rangle|\le t} |\langle x,\theta\rangle|^b d\bar{\mu}(x)$. Then $\bar{V}_a(t,\theta) = V_a(t, T^*\theta)$ and $\bar{U}_b(t,\theta) = U_b(t, T^*\theta)$. So if $\bar{\alpha}(\cdot)$ is the index function corresponding to $\bar{\mu}$, we set $\alpha(\theta) = \bar{\alpha}((T^*)^{-1}\theta)$ and obtain the index function for the given μ. Hence we can (and will) assume without loss of generality that (B_n) is block diagonal with respect to $V_1 \oplus \cdots \oplus V_p$. This is the first step in our reduction argument.

Next we show that we can (and hence will) assume without loss of generality that the spaces $V_1, \dots, V_p$ are mutually orthogonal. In fact let $(\cdot,\cdot)$ be an inner product on $\mathbb{R}^d$ which makes the V_i-spaces mutually orthogonal. Let $\tilde{V}_a(t,\theta) = \int_{|(x,\theta)|>t} |(x,\theta)|^a d\mu(x)$ and $\tilde{U}_b(t,\theta) = \int_{|(x,\theta)|\le t} |(x,\theta)|^b d\mu(x)$. Since $(x,\theta) = \langle x, Q\theta\rangle$ for some symmetric positive definite matrix Q we have $\tilde{V}_a(t,\theta) = V_a(t,Q\theta)$ and $\tilde{U}_b(t,\theta) = U_b(t,Q\theta)$. Hence if $\tilde{\alpha}(\cdot)$ is the index function of μ for the functions $\tilde{V}_a$ and $\tilde{U}_b$ we can set $\alpha(\theta) = \tilde{\alpha}(Q^{-1}\theta)$ and obtain an index function for V_a and U_b, respectively.

In the following we assume that B_n is block diagonal with respect to $V_1 \oplus \cdots \oplus V_p$ where the spaces $V_1, \dots, V_p$ are mutually orthogonal with respect to $\langle\cdot,\cdot\rangle$.

We need some more notation. For $i = 1, \dots, p$ let $\pi_i : \mathbb{R}^d \to V_i$ be the orthogonal projection. Projecting (2.6) onto V_i yields $\mu_i = \pi_i(\mu) \in \mathrm{ROV}(F_i, c)$. For $\theta \in V_i$ let

$$V_a^{(i)}(t,\theta) = \int_{|\langle x,\theta\rangle|>t} |\langle x,\theta\rangle|^a d\mu_i(x)$$

and

$$U_b^{(i)}(t,\theta) = \int_{|\langle x,\theta\rangle|\le t} |\langle x,\theta\rangle|^b d\mu_i(x).$$

Furthermore we write $x \in \mathbb{R}^d$ as $x = x^{(1)} + \cdots + x^{(p)}$ with respect to the direct sum decomposition.

The crucial technical point in deriving bounds on V_a and U_b for a fixed θ is the following Proposition which relates the asymptotic behavior of V_a and U_b to that of $V_a^{(i)}$ and $U_b^{(i)}$.

Proposition 3.1. *Let $\mu \in \mathrm{ROV}(F,c)$ for some $c > 1$ and let $\mathbb{R}^d = V_1 \oplus \cdots \oplus V_p$ be the spectral decomposition relative to $F = F_1 \oplus \cdots \oplus F_p$. Assume further that $V_1, \dots, V_p$ are mutually orthogonal and that the norming operators for μ in* (2.6) *are block diagonal with respect to that decomposition. Then if $i = 1, \dots, p$ and $\theta \in V_1 \oplus \cdots \oplus V_i \setminus V_1 \oplus \cdots \oplus V_{i-1}$ we have for $a < \frac{1}{a_i} < b$:*

(a) *There exist constants $C_1, C_2 > 0$ and a $t_1 \ge 1$ such that*

$$C_1 t^{a-b} U_b^{(i)}(t,\theta^{(i)}) \le V_a(t,\theta) \le C_2 V_a^{(i)}(t,\theta^{(i)}) \tag{3.1}$$

whenever $t \ge t_1$. Especially $V_a(t,\theta)$ exists.

(b) *There exist constants $C_3, C_4 > 0$ and a $t_2 \ge 1$ such that*

$$C_3 U_b^{(i)}(t,\theta^{(i)}) \le U_b(t,\theta) \le C_4 t^{b-a} V_a^{(i)}(t,\theta^{(i)}) \tag{3.2}$$

whenever $t \ge t_2$.

Proof. The proof of Proposition 3.1 is quite involved and technical. As in section 4 of [13] we have to argue in several steps. We first show the upper bound in

(3.1). This together with a variant of (4.24) of [13] then implies the upper bound in (3.2). The proof of the lower bounds is even more complicated. We first have to show the lower bound in (3.2) if $\frac{1}{a_i} < b < \frac{1}{a_{i-1}}$. Then we use this estimate together with the upper bound in (3.1) to show (3.2) in general. Finally, after proving a variant of (4.23) of [13] the lower bound in (3.1) follows.

We start with the upper bound in (3.1). Write $\theta = \theta^{(1)} + \cdots + \theta^{(i)}$ and note that $\theta^{(i)} \neq 0$. Let $L_i = V_1 \oplus \cdots \oplus V_i$ and $P_i : \mathbb{R}^d \to L_i$ the orthogonal projection. Projecting (2.6) onto L_i yields $\bar{\mu}_i = P_i(\mu) \in \mathrm{ROV}(\bar{F}_i, c)$ where $\bar{F}_i = F_1 \oplus \cdots \oplus F_i$ has real spectrum $0 < a_1 < \cdots < a_i$. For $\theta \in L_i$ define

$$\bar{V}_a^{(i)}(t,\theta) = \int_{|\langle x,\theta\rangle|>t} |\langle x,\theta\rangle|^a d\bar{\mu}_i(x)$$

and

$$\bar{U}_b^{(i)}(t,\theta) = \int_{|\langle x,\theta\rangle|\leq t} |\langle x,\theta\rangle|^b d\bar{\mu}_i(x).$$

Then for $\theta \in L_i$ we have $V_a(t,\theta) = \bar{V}_a^{(i)}(t,\theta)$ and $U_b(t,\theta) = \bar{U}_b^{(i)}(t,\theta)$. Therefore by Theorem 4.16 of [13] $\bar{V}_a$ and hence V_a exists for $a < \frac{1}{a_i}$.

Using the well known inequality $|x+y|^a \leq 2^a(|x|^a + |y|^a)$ for $x, y \in \mathbb{R}$ we can bound

$$\begin{aligned}
V_a(t,\theta) &= \int_{|\langle x,\theta\rangle|>t} \big|\langle x^{(i)},\theta^{(i)}\rangle + \langle P_{i-1}(x), P_{i-1}(\theta)\rangle|^a d\mu(x) \\
&\leq 2^a\Big(\int_{|\langle x,\theta\rangle|>t} |\langle x^{(i)},\theta^{(i)}\rangle|^a d\mu(x) + \int_{|\langle x,\theta\rangle|>t} |\langle P_{i-1}(x), P_{i-1}(\theta)\rangle|^a d\mu(x)\Big) \\
&= 2^a(I_1(t) + I_2(t)).
\end{aligned}$$

But $\{x : |\langle x,\theta\rangle| > t\} \subset \{x : |\langle x^{(i)},\theta^{(i)}\rangle| > t/2\} \cup \{x : |\langle P_{i-1}(x), P_{i-1}(\theta)\rangle| > t/2\}$ and hence we can decompose further to obtain

$$\begin{aligned}
I_1(t) &\leq \int_{|\langle x^{(i)},\theta^{(i)}\rangle|>t/2} |\langle x^{(i)},\theta^{(i)}\rangle|^a d\mu(x) + \int_{|\langle P_{i-1}(x),P_{i-1}(\theta)\rangle|>t/2} |\langle x^{(i)},\theta^{(i)}\rangle|^a d\mu(x) \\
&= V_a^{(i)}(\frac{t}{2},\theta^{(i)}) + \int\limits_{\substack{|\langle P_{i-1}(x),P_{i-1}(\theta)\rangle|>t/2 \\ |\langle x^{(i)},\theta^{(i)}\rangle|\leq t/2}} |\langle x^{(i)},\theta^{(i)}\rangle|^a d\mu(x) \\
&\qquad + \int\limits_{\substack{|\langle P_{i-1}(x),P_{i-1}(\theta)\rangle|>t/2 \\ |\langle x^{(i)},\theta^{(i)}\rangle|>t/2}} |\langle x^{(i)},\theta^{(i)}\rangle|^a d\mu(x) \\
&\leq 2V_a^{(i)}(\frac{t}{2},\theta^{(i)}) + \big(\frac{t}{2}\big)^a \bar{V}_0^{(i-1)}(\frac{t}{2}, P_{i-1}(\theta)).
\end{aligned}$$

On the other hand

$$
\begin{aligned}
I_2(t) &\le \int_{|\langle P_{i-1}(x),P_{i-1}(\theta)\rangle|>t/2} |\langle P_{i-1}(x),P_{i-1}(\theta)\rangle|^a d\mu(x)\\
&\quad + \int_{|\langle x^{(i)},\theta^{(i)}\rangle|>t/2} |\langle P_{i-1}(x),P_{i-1}(\theta)\rangle|^a d\mu(x)\\
&= \bar V_a^{(i-1)}(\frac{t}{2},P_{i-1}(\theta)) + \int\limits_{\substack{|\langle x^{(i)},\theta^{(i)}\rangle|>t/2\\ |\langle P_{i-1}(x),P_{i-1}(\theta)\rangle|\le t/2}} |\langle P_{i-1}(x),P_{i-1}(\theta)\rangle|^a d\mu(x)\\
&\qquad + \int\limits_{\substack{|\langle x^{(i)},\theta^{(i)}\rangle|>t/2\\ |\langle P_{i-1}(x),P_{i-1}(\theta)\rangle|> t/2}} |\langle P_{i-1}(x),P_{i-1}(\theta)\rangle|^a d\mu(x)\\
&\le 2\bar V_a^{(i-1)}(\frac{t}{2},P_{i-1}(\theta)) + \big(\frac{t}{2}\big)^a V_0^{(i)}(\frac{t}{2},\theta^{(i)}).
\end{aligned}
$$

Putting things together it follows

$$
\begin{aligned}
V_a(t,\theta) \le 2^a\Big(& 2V_a^{(i)}(\frac{t}{2},\theta^{(i)}) + \big(\frac{t}{2}\big)^a V_0^{(i)}(\frac{t}{2},\theta^{(i)})\\
&+ 2\bar V_a^{(i-1)}(\frac{t}{2},P_{i-1}(\theta)) + \big(\frac{t}{2}\big)^a \bar V_0^{(i-1)}(\frac{t}{2},P_{i-1}(\theta))\Big).
\end{aligned}
\tag{3.3}
$$

Now we need a result which is independent interest.

Lemma 3.2. *Suppose $\mu \in \mathrm{ROV}(F,c)$ for some $c > 1$ where $0 < a_1 < \cdots < a_p$ denote the real parts of the eigenvalues of F. Then for $a < \frac{1}{a_p}$ and any compact $K \subset \Gamma$ there exist constants $A_1, A_2 > 0$ and a $t_2 \ge 1$ such that*

$$A_1 t^a V_0(t,\theta) \le V_a(t,\theta) \le A_2 t^a V_0(t,\theta)$$

whenever $t \ge t_2$ and $\theta \in K$.

Proof. For $b > \frac{1}{a_1}$ write

$$t^a V_0(t,\theta) = \frac{t^b V_0(t,\theta)}{U_b(t,\theta)} \cdot \frac{U_b(t,\theta)}{t^{b-a} V_a(t,\theta)} V_a(t,\theta)$$

and apply Theorem 4.20 of [13] to obtain the lower bound. To prove the upper bound write

$$V_a(t,\theta) = \frac{t^{b-a} V_a(t,\theta)}{U_b(t,\theta)} \cdot \frac{U_b(t,\theta)}{t^b V_0(t,\theta)} t^a V_0(t,\theta)$$

and use Theorem 4.20 of [13] again. □

Now since $\mu_i \in \mathrm{ROV}(F_i,c)$ and $P_{i-1}(\mu) \in \mathrm{ROV}(F_1 \oplus \cdots \oplus F_{i-1},c)$ and $a < \frac{1}{a_i} < \frac{1}{a_{i-1}}$ it follows from Lemma 3.2 and (3.3) that for some constants $K_1, K_2 > 0$ and some $t_0 \geq 1$ we can bound

$$V_a(t,\theta) \leq K_1 V_a^{(i)}(\frac{t}{2},\theta^{(i)}) + K_2 \bar{V}_a^{(i-1)}(\frac{t}{2},P_{i-1}(\theta)) \tag{3.4}$$

whenever $t \geq t_0$. On the other hand it follows from Corollary 4.21(a)(ii) of [13] that for $\delta > 0$ (sufficiently small) there exist constants $C_1, C_2 > 0$ such that for all large t

$$\frac{\bar{V}_a^{(i-1)}(\frac{t}{2},P_{i-1}(\theta))}{V_a^{(i)}(\frac{t}{2},\theta^{(i)})} \leq \frac{C_1 (t/2)^{a-\frac{1}{a_{i-1}}+\delta}}{C_2 (t/2)^{a-\frac{1}{a_i}-\delta}} = K t^{\frac{1}{a_i}-\frac{1}{a_{i-1}}+2\delta} \tag{3.5}$$

which tends to zero as $t \to \infty$ if $\delta > 0$ is chosen small enough. Hence by (3.4) and Corollary 4.21(a)(i) of [13] there exists a $t_1 \geq t_0$ and a constant $M > 0$ such that for $t \geq t_1$

$$\begin{aligned} V_a(t,\theta) &\leq M V_a^{(i)}(\frac{t}{2},\theta^{(i)}) \\ &= M \left(\frac{V_a^{(i)}(2\cdot\frac{t}{2},\theta^{(i)})}{V_a^{(i)}(\frac{t}{2},\theta^{(i)})} \right)^{-1} V_a^{(i)}(t,\theta^{(i)}) \\ &\leq \bar{M} V_a^{(i)}(t,\theta^{(i)}) \end{aligned}$$

for some constant $\bar{M} > 0$ which proves the upper bound in (3.1).

Lemma 3.3. *Under the assumptions of Proposition 3.1 there exists a constant $K_1 > 0$ such that*

$$\limsup_{t\to\infty} \frac{U_b(t,\theta)}{t^{b-a} V_a^{(i)}(t,\theta^{(i)})} \leq K_1.$$

Proof. Fix any $0 < \lambda < 1$. As in the proof of Theorem 4.8 of [13] it follows that

$$\frac{U_b(t,\theta)}{t^{b-a} V_a^{(i)}(t,\theta^{(i)})} \leq -\frac{V_a(t,\theta)}{V_a^{(i)}(t,\theta^{(i)})} + (\lambda^{-(b-a)} - 1) \sum_{n=1}^{\infty} \lambda^{n(b-a)} \frac{V_a(\lambda^n t,\theta)}{V_a^{(i)}(t,\theta^{(i)})}.$$

But by the upper bound in (3.1) together with Theorem 4.16 of [13] we obtain for any $\delta > 0$ constants $C_2, C_3, \bar{C} > 0$ such that

$$\begin{aligned} \limsup_{t\to\infty} \frac{V_a(\lambda^n t,\theta)}{V_a^{(i)}(t,\theta^{(i)})} &\leq C_2 \limsup_{t\to\infty} \frac{V_a^{(i)}(\lambda^n t,\theta^{(i)})}{V_a^{(i)}(t,\theta^{(i)})} \\ &= C_2 \left(\liminf_{t\to\infty} \frac{V_a^{(i)}(\lambda^{-n}(\lambda^n t),\theta^{(i)})}{V_a^{(i)}(\lambda^n t,\theta^{(i)})} \right)^{-1} \\ &\leq C_2 \Big(C_3 \big(\lambda^{-n}\big)^{a-\frac{1}{a_i}-\delta} \Big)^{-1} \\ &= \bar{C} \lambda^{n(a-\frac{1}{a_i}-\delta)}. \end{aligned}$$

Hence

$$\limsup_{t\to\infty} \frac{U_b(t,\theta)}{t^{b-a}V_a^{(i)}(t,\theta^{(i)})} \leq -C_2 + (\lambda^{-(b-a)} - 1)\bar{C}\sum_{n=1}^{\infty} \lambda^{n(b-\frac{1}{a_i}-\delta)}$$

which is finite since $b > \frac{1}{a_i}$ and $\delta > 0$ can be made arbitrary small. This concludes the proof of Lemma 3.3. □

Since we can write

$$U_b(t,\theta) = \frac{U_b(t,\theta)}{t^{b-a}V_a^{(i)}(t,\theta^{(i)})} t^{b-a}V_a^{(i)}(t,\theta^{(i)})$$

the upper bound in (3.2) follows immediately from Lemma 3.3.

Now we show the lower bound in (3.2). Assume first that $\frac{1}{a_i} < b < \frac{1}{a_{i-1}}$. It follows from $|x+y|^b \leq 2^b(|x|^b + |y|^b)$ that $|x+y|^b \geq 2^{-b}|x|^b - |y|^b$ and hence

(3.6)

$$\begin{aligned} U_b(t,\theta) &= \int_{|\langle x,\theta\rangle|\leq t} |\langle x^{(i)},\theta^{(i)}\rangle + \langle P_{i-1}(x), P_{i-1}(\theta)\rangle|^b d\mu(x) \\ &\geq 2^{-b}\int_{|\langle x,\theta\rangle|\leq t} |\langle x^{(i)},\theta^{(i)}\rangle|^b d\mu(x) - \int_{|\langle x,\theta\rangle|\leq t} |\langle P_{i-1}(x), P_{i-1}(\theta)\rangle|^b d\mu(x) \\ &= 2^{-b}J_1(t) - J_2(t). \end{aligned}$$

Then for $\gamma > 0$ (which we will specify later) we decompose further

$$\begin{aligned} J_1(t) &= \int_{\substack{|\langle x,\theta\rangle|\leq t \\ |\langle x^{(i)},\theta^{(i)}\rangle|\leq \gamma t}} |\langle x^{(i)},\theta^{(i)}\rangle|^b d\mu(x) + \int_{\substack{|\langle x,\theta\rangle|\leq t \\ |\langle x^{(i)},\theta^{(i)}\rangle|> \gamma t}} |\langle x^{(i)},\theta^{(i)}\rangle|^b d\mu(x) \\ &\geq \int_{\substack{|\langle x,\theta\rangle|\leq t \\ |\langle x^{(i)},\theta^{(i)}\rangle|\leq \gamma t}} |\langle x^{(i)},\theta^{(i)}\rangle|^b d\mu(x) \\ &= \int_{|\langle x^{(i)},\theta^{(i)}\rangle|\leq \gamma t} |\langle x^{(i)},\theta^{(i)}\rangle|^b d\mu(x) - \int_{\substack{|\langle x,\theta\rangle|> t \\ |\langle x^{(i)},\theta^{(i)}\rangle|\leq \gamma t}} |\langle x^{(i)},\theta^{(i)}\rangle|^b d\mu(x) \\ &= U_b^{(i)}(\gamma t, \theta^{(i)}) - J_3(t). \end{aligned}$$

But $J_3(t) \leq (\gamma t)^b \bar{V}_0^{(i)}(t,\theta)$ and hence for any $\gamma > 0$ and any $t > 0$

$$J_1(t) \geq U_b^{(i)}(\gamma t, \theta^{(i)}) - (\gamma t)^b \bar{V}_0^{(i)}(t,\theta). \tag{3.7}$$

Using $\{|\langle x,\theta\rangle| > t\} \subset \{|\langle x^{(i)},\theta^{(i)}\rangle| > t/2\} \cup \{|\langle P_{i-1}(x), P_{i-1}(\theta)\rangle| > t/2\}$ again, we can bound

$$\bar{V}_0^{(i)}(t,\theta) \leq V_0^{(i)}(\frac{t}{2},\theta^{(i)}) + \bar{V}_0^{(i-1)}(\frac{t}{2}, P_{i-1}(\theta))$$

and by (3.5) $\bar{V}_0^{(i-1)}(t, P_{i-1}(\theta))/V_0^{(i)}(t, \theta^{(i)}) \to 0$ as $t \to \infty$. Then there exists a $t_0 \geq 1$ such that $\bar{V}_0^{(i)}(t, \theta) \leq \frac{3}{2} V_0^{(i)}(\frac{t}{2}, \theta^{(i)})$ whenever $t \geq t_0$. Hence by (3.7) we have for any $\gamma > 0$ and $t \geq t_0$

$$J_1(t) \geq U_b^{(i)}(\gamma t, \theta^{(i)}) - (\gamma t)^b V_0^{(i)}(\frac{t}{2}, \theta^{(i)}). \tag{3.8}$$

Apply Theorem 4.9 and Theorem 4.14 of [13] to obtain constants $C, \bar{C} > 0$ and a $\bar{t}_0 \geq t_0$ such that for $t \geq \bar{t}_0$

$$\begin{aligned} t^b V_0^{(i)}(\frac{t}{2}, \theta^{(i)}) &= 2^{-b} \frac{(t/2)^b V_0^{(i)}(t/2, \theta^{(i)})}{U_b^{(i)}(t/2, \theta^{(i)})} U_b^{(i)}(t/2, \theta^{(i)}) \\ &\leq C U_b^{(i)}(t/2, \theta^{(i)}) \\ &\leq \bar{C} U_b^{(i)}(t, \theta^{(i)}). \end{aligned}$$

Hence in view of (3.8) we have shown that there exists a constant $\tilde{C} > 0$ and a $t_* \geq 1$ such that for any $\gamma > 0$

$$J_1(t) \geq U_b^{(i)}(\gamma t, \theta^{(i)}) - \gamma^b \tilde{C} U_b^{(i)}(t, \theta^{(i)}) \tag{3.9}$$

whenever $t \geq t_*$. Now choose $\delta > 0$ such that $-1/a_i + \delta < 0$. Then by Theorem 4.9 of [13] there exists a constant $C_2 > 0$ such that for any $0 < \gamma < 1$ there exists a $t_1(\gamma) \geq t_*$ with

$$\frac{U_b^{(i)}(\gamma t, \theta^{(i)})}{U_b^{(i)}(t, \theta^{(i)})} = \left(\frac{U_b^{(i)}(\gamma^{-1}(\gamma t), \theta^{(i)})}{U_b^{(i)}(\gamma t, \theta^{(i)})} \right)^{-1} \geq C_2 \gamma^{-\frac{1}{a_i} + b + \delta}$$

whenever $t \geq t_1(\gamma)$. Then (3.9) yields for any $0 < \gamma < 1$ and $t \geq t_1(\gamma)$

$$J_1(t) \geq \gamma^b \big(C_2 \gamma^{-\frac{1}{a_i} + \delta} - \tilde{C} \big) U_b^{(i)}(t, \theta^{(i)}).$$

Now choose $0 < \gamma_0 < 1$ such that $C_2 \gamma_0^{-1/a_i + \delta} \geq 2\tilde{C}$. Then we have shown that there exists a constant $D > 0$ and some $t_2 \geq 1$ such that

$$J_1(t) \geq D U_b^{(i)}(t, \theta^{(i)}) \tag{3.10}$$

whenever $t \geq t_2$.

Next we bound J_2 from above. Note that

$$\begin{aligned} J_2(t) = & \int\limits_{\substack{|\langle x, \theta \rangle| \leq t \\ |\langle P_{i-1}(x), P_{i-1}(\theta) \rangle| \leq t}} |\langle P_{i-1}(x), P_{i-1}(\theta) \rangle|^b d\mu(x) \\ & + \int\limits_{\substack{|\langle x, \theta \rangle| \leq t \\ |\langle P_{i-1}(x), P_{i-1}(\theta) \rangle| > t}} |\langle P_{i-1}(x), P_{i-1}(\theta) \rangle|^b d\mu(x) \\ & \leq \bar{U}_b^{(i-1)}(t, P_{i-1}(\theta)) + \bar{V}_b^{(i-1)}(t, P_{i-1}(\theta)). \end{aligned}$$

Since $1/a_i < b < 1/a_{i-1}$ it follows from Theorem 4.16 of [13] that $\bar{V}_b^{(i-1)}(t, P_{i-1}(\theta))$ exists and hence $\lim_{t\to\infty} \bar{U}_b^{(i-1)}(t, P_{i-1}(\theta)) = M < \infty$. Therefore, by Corollary 4.21 of [13] for any $\delta > 0$ sufficiently small there exist constants $C_2, B_3 > 0$ such that as $t \to \infty$

$$\begin{aligned} \frac{J_2(t)}{U_b^{(i)}(t,\theta^{(i)})} &\leq \frac{M}{U_b^{(i)}(t,\theta^{(i)})} + \frac{\bar{V}_b^{(i-1)}(t, P_{i-1}(\theta))}{U_b^{(i)}(t,\theta^{(i)})} \\ &\leq \frac{M}{B_3 t^{b-1/a_i-\delta}} + \frac{C_2 t^{b-1/a_{i-1}+\delta}}{B_3 t^{b-1/a_i-\delta}} \\ &\to 0. \end{aligned}$$

Therefore, given $\varepsilon > 0$ there exists a $t_3(\varepsilon) \geq 1$ such that for all $t \geq t_3(\varepsilon)$

$$J_2(t) \leq \varepsilon U_b^{(i)}(t,\theta^{(i)}). \tag{3.11}$$

This together with (3.10) and (3.6) proves the lower bound in (3.2) for $1/a_i < b < 1/a_{i-1}$.

If $1/a_i < 1/a_{i-1} \leq b$ we have to argue differently. Write $b = b_1 + r_1$ for some $1/a_i < b_1 < 1/a_{i-1}$ and some $r_1 > 0$. Then for any $0 < \gamma < 1$ we obtain

$$\begin{aligned} U_b(t,\theta) &\geq \int_{\gamma t < |\langle x,\theta\rangle| \leq t} |\langle x,\theta\rangle|^b d\mu(x) \\ &\geq (\gamma t)^{r_1} \int_{\gamma t < |\langle x,\theta\rangle| \leq t} |\langle x,\theta\rangle|^b d\mu(x) \\ &= (\gamma t)^{r_1} U_{b_1}(t,\theta) \left(1 - \frac{U_{b_1}(\gamma t,\theta)}{U_{b_1}(t,\theta)}\right). \end{aligned} \tag{3.12}$$

Using the already proved bounds on U_{b_1} in (3.2) we get for $a < 1/a_i$, some constants $C_3, C_4 > 0$ and some $t_2 \geq 1$ that

$$\begin{aligned} \frac{U_{b_1}(\gamma t,\theta)}{U_{b_1}(t,\theta)} &\leq \frac{C_4 (\gamma t)^{b_1-a} V_a^{(i)}(\gamma t,\theta^{(i)})}{C_3 U_{b_1}^{(i)}(t,\theta^{(i)})} \\ &= C\gamma^{b_1-a} \frac{t^{b_1-a} V_a^{(i)}(t,\theta^{(i)})}{U_{b_1}^{(i)}(t,\theta^{(i)})} \cdot \frac{V_a^{(i)}(\gamma t,\theta^{(i)})}{V_a^{(i)}(t,\theta^{(i)})}. \end{aligned}$$

Hence by Theorem 4.20 and Theorem 4.16 of [13] for $\delta > 0$ sufficiently small there exists a constant $D > 0$ such that for any $0 < \gamma < 1$ there exists a $t_2(\gamma) \geq t_2$ with

$$\frac{U_{b_1}(\gamma t,\theta)}{U_{b_1}(t,\theta)} \leq D\gamma^{b_1 - \frac{1}{a_i} - \delta} \quad \text{whenever } t \geq t_2(\gamma).$$

Now choose $0 < \gamma_0 < 1$ so that $D\gamma_0^{b_1-1/a_i-\delta} < 1/2$. Then for all $t \geq t_2(\gamma_0)$ it follows from (3.12) that

$$U_b(t,\theta) \geq \frac{1}{2}(\gamma_0 t)^{r_1} U_{b_1}(t,\theta)$$

and hence by the already proved bound for U_{b_1} we conclude that

$$U_b(t,\theta) \geq C t^{r_1} U_{b_1}^{(i)}(t,\theta^{(i)})$$

whenever $t \geq \bar{t}_2$, some $\bar{t}_2 \geq t_2(\gamma_0)$ and some constant $C > 0$. Now an application of Theorem 4.20 of [13] yields

$$\begin{aligned} t^{r_1} U_{b_1}^{(i)}(t,\theta^{(i)}) &= \frac{U_{b_1}^{(i)}(t,\theta^{(i)})}{t^{b_1} V_0^{(i)}(t,\theta^{(i)})} \cdot \frac{t^{b_1+r_1} V_0^{(i)}(t,\theta^{(i)})}{U_{b_1}^{(i)}(t,\theta^{(i)})} \cdot U_b^{(i)}(t,\theta^{(i)}) \\ &\geq \tilde{C} U_b^{(i)}(t,\theta^{(i)}) \end{aligned}$$

for all large t and some constant $\tilde{C} > 0$. This proves the lower bound in (3.2) in the general case.

To prove the lower bound in (3.1) we need another technical result.

Lemma 3.4. *Under the assumptions of Proposition 3.1 there exists a constant* $K_2 > 0$ *such that*

$$\liminf_{t\to\infty} \frac{t^{b-a} V_a(t,\theta)}{U_b^{(i)}(t,\theta^{(i)})} \geq K_2.$$

Proof. Fix any $\lambda > 1$. As in the proof of Theorem 4.14 of [13] it follows that

$$V_a(t,\theta) \geq -(\lambda t)^{a-b} U_b(t,\theta) + (1-\lambda^{a-b}) t^{a-b} \sum_{n=1}^{\infty} \lambda^{n(a-b)} U_b(\lambda^n t,\theta)$$

and hence by the lower bound in (3.2) we obtain a constant $C_3 > 0$ and a $t_2 \geq 1$ such that

$$\frac{t^{b-a} V_a(t,\theta)}{U_b^{(i)}(t,\theta^{(i)})} \geq -\lambda^{a-b} C_3 + (1-\lambda^{a-b}) C_3 \sum_{n=1}^{\infty} \lambda^{n(a-b)} \frac{U_b^{(i)}(\lambda^n t,\theta^{(i)})}{U_b^{(i)}(t,\theta^{(i)})}.$$

But by Corollary 4.21 of [13], for any $\delta > 0$ there exists another constant $C_4 > 0$ and a $\bar{t}_2 \geq t_2$ such that for all $n \geq 1$

$$\frac{U_b^{(i)}(\lambda^n t,\theta^{(i)})}{U_b^{(i)}(t,\theta^{(i)})} \geq C_4 \lambda^{n(b-\frac{1}{a_i}-\delta)}$$

whenever $t \geq \bar{t}_2$. Hence for $t \geq \bar{t}_2$

$$\frac{t^{b-a} V_a(t,\theta)}{U_b^{(i)}(t,\theta^{(i)})} \geq -\lambda^{a-b} C_3 + (1-\lambda^{a-b}) C_3 C_4 \sum_{n=1}^{\infty} \lambda^{n(a-\frac{1}{a_i}-\delta)}$$

and the last series converges since $a < 1/a_i$ and is greater zero if $\lambda > 1$ is chosen large enough. □

Now if we write

$$V_a(t,\theta) = \frac{t^{b-a}V_a(t,\theta)}{U_b^{(i)}(t,\theta^{(i)})} t^{a-b} U_b^{(i)}(t,\theta^{(i)})$$

the lower bound in (3.1) is immediate from Lemma 3.4 and the proof of Proposition 3.1 is complete. □

We are now in position to state the main result of this section. With the aid of Proposition 3.1 the proof will be fairly straight forward.

Theorem 3.5. *Suppose that $\mu \in \mathrm{ROV}(F, c)$ for some $c > 0$ where all the real parts of the eigenvalues of F are positive. Then there exists an index function $\alpha : \Gamma \to \mathbb{R}_+$ such that for any $\theta \in \Gamma$ and $\delta > 0$ we have:*

(a) *If $a < \frac{1}{\alpha(\theta)}$ then $V_a(t, \theta)$ exists and*

  (i) *there exist constants $C_1, C_2 > 0$ and a $t_1 \geq 1$ such that*

$$C_1 \lambda^{a - \frac{1}{\alpha(\theta)} - \delta} \leq \frac{V_a(\lambda t, \theta)}{V_a(t, \theta)} \leq C_2 \lambda^{a - \frac{1}{\alpha(\theta)} + \delta}$$

  whenever $t \geq t_1$ and $\lambda \geq 1$.

  (ii) *there exist constants $B_1, B_2 > 0$ and a $\bar{t}_1 \geq 1$ such that*

$$B_1 t^{a - \frac{1}{\alpha(\theta)} - \delta} \leq V_a(t, \theta) \leq B_2 t^{a - \frac{1}{\alpha(\theta)} + \delta}$$

  whenever $t \geq \bar{t}_1$.

(b) *If $b > \frac{1}{\alpha(\theta)}$ then*

  (i) *there exist constants $C_3, C_4 > 0$ and a $t_2 \geq 1$ such that*

$$C_3 \lambda^{b - \frac{1}{\alpha(\theta)} - \delta} \leq \frac{U_b(\lambda t, \theta)}{U_b(t, \theta)} \leq C_4 \lambda^{b - \frac{1}{\alpha(\theta)} + \delta}$$

  whenever $t \geq t_2$ and $\lambda \geq 1$.

  (ii) *there exist constants $B_3, B_4 > 0$ and a $\bar{t}_2 \geq 1$ such that*

$$B_3 t^{b - \frac{1}{\alpha(\theta)} - \delta} \leq U_b(t, \theta) \leq B_4 t^{b - \frac{1}{\alpha(\theta)} + \delta}$$

  whenever $t \geq \bar{t}_2$.

Proof. We only prove the first assertions in (a) and (b). Part (ii) of (a) and (b) follows from the corresponding results in part (i) as in the proof of Corollary 4.15 of [13]. As pointed out at the beginning of section 3 we can assume without loss of generality that μ satisfies the assumptions of Proposition 3.1. Then if $\theta \in \Gamma$ there exists a unique $i = 1, \dots, p$ such that $\theta \in V_1 \oplus \cdots \oplus V_i \setminus V_1 \oplus \cdots \oplus V_{i-1}$. We define $\alpha(\theta) = a_i$.

By Proposition 3.1 together with Theorem 4.20 and Theorem 4.9 of [13] we have for all large t and any $\lambda \geq 1$

$$\begin{aligned} \frac{V_a(\lambda t, \theta)}{V_a(t, \theta)} &\leq \frac{C_2 V_a^{(i)}(\lambda t, \theta^{(i)})}{C_1 t^{a-b} U_b^{(i)}(t, \theta^{(i)})} \\ &= \lambda^{a-b} \frac{C_2}{C_1} \frac{(\lambda t)^{b-a} V_a^{(i)}(\lambda t, \theta^{(i)})}{U_b^{(i)}(\lambda t, \theta^{(i)})} \frac{U_b^{(i)}(\lambda t, \theta^{(i)})}{U_b^{(i)}(t, \theta^{(i)})} \\ &\leq C \lambda^{a - \frac{1}{a_i} + \delta} = C \lambda^{a - \frac{1}{\alpha(\theta)} + \delta} \end{aligned}$$

for some constant $C > 0$ independent of λ.

Similarly, for t large

$$\begin{aligned}\frac{V_a(\lambda t,\theta)}{V_a(t,\theta)} &\geq \frac{C_1(\lambda t)^{a-b}U_b^{(i)}(\lambda t,\theta^{(i)})}{C_2 V_a^{(i)}(t,\theta^{(i)})}\\ &= \frac{C_1}{C_2}\frac{(\lambda t)^{a-b}U_b^{(i)}(\lambda t,\theta^{(i)})}{V_a^{(i)}(\lambda t,\theta^{(i)})}\frac{V_a^{(i)}(\lambda t,\theta^{(i)})}{V_a^{(i)}(t,\theta^{(i)})}\\ &\geq \bar{C}\lambda^{a-\frac{1}{\alpha(\theta)}-\delta}\end{aligned}$$

for some constant $\bar{C} > 0$ independent of λ. The proof of the first assertion in (b) is similar, utilizing Proposition 3.1 and the bounds on $U_b^{(i)}$ and therefore omitted. This concludes the proof. □

Complementary to Theorem 4.20 of [13] the following refined version of Karamata bounds on the ratio between U_b and V_a holds true.

Theorem 3.6. *Suppose $\mu \in \mathrm{ROV}(F, c)$ for some $c > 1$ where all the real parts of the eigenvalues of F are positive. Then for the index function $\alpha(\cdot)$ obtained in Theorem 3.5 we get for any $\theta \in \Gamma$ and $a < 1/\alpha(\theta) < b$ constants $K_1, K_2 > 0$ such that*

$$\limsup_{t\to\infty} \frac{t^{b-a}V_a(t,\theta)}{U_b(t,\theta)} \leq K_1$$

and

$$\limsup_{t\to\infty} \frac{U_b(t,\theta)}{t^{b-a}V_a(t,\theta)} \leq K_2.$$

Proof. The proof is identical to the proof of Theorem 4.20 of [13] utilizing Theorem 3.5. See also the proof of Theorem 4.8 and Theorem 4.14 of [13]. □

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