HAUSDORFF DIMENSION OF OPERATOR STABLE SAMPLE PATHS

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Abstract. The Hausdorff dimension of the sample paths of a stochastic process with stationary independent operator stable increments is computed. With probability one, every sample path has the same dimension, depending on the real parts of the eigenvalues of the operator stable exponent.

1. Introduction

Operator stable laws are a natural multivariable analogue of one dimensional stable laws [8]. They are useful in geophysics [12] and finance [13] to model heavy tailed random vectors with different tail behavior in each coordinate. Operator stable Lévy motions are scaling limits of simple random walks on $\mathbb{R}^d$, normalized by linear operators [11]. The sample paths of a stable Lévy motion on $\mathbb{R}^d$ for $d \geq 2$ are random fractals whose dimension α equals the stable index [19]. In this paper, we begin extending this result to operator Lévy motions, by computing the Hausdorff dimension of the sample paths.

If $\{X_t\}$ is an α-stable Lévy motion on $\mathbb{R}^d$ then Blumenthal and Getoor [5] show that $\dim X_{[0,1]} = \min(\alpha, d)$ with probability one, where the sample path $X_{[0,1]}(\omega) = \{X_t(\omega) : 0 \leq t \leq 1\}$ for any element ω of the sample space. Pruitt and Taylor [16] compute $\dim X_{[0,1]}$ for an operator Lévy motion $X_t = (X_t^{(1)}, \ldots, X_t^{(d)})$ where the ith marginal process $\{X_t^{(i)}\}$ is an α_i-stable Lévy motion on $\mathbb{R}^1$, $\alpha_1 \geq \cdots \geq \alpha_d$, and $\{X_t^{(i)}\}$ is independent of $\{X_t^{(j)}\}$ for $i \neq j$. If $\alpha_1 \leq 1$ or if $\alpha_1 = \alpha_2$ then $\dim X_{[0,1]} = \alpha_1$ almost surely, and otherwise $\dim X_{[0,1]} = \rho$ almost surely where $\rho = 1 + \alpha_2(1 - 1/\alpha_1)$ so that $\alpha_1 > \rho > \alpha_2$. In this paper, we compute $\dim X_{[0,1]}$ for an arbitrary operator Lévy motion on $\mathbb{R}^d$. This includes the case of a process with stable marginals, but without assuming independence.

2. Results

A probability measure ν on $\mathbb{R}^d$ is *full* if it cannot be supported on any $d-1$ dimensional hyperplane. A full infinitely divisible probability measure ν on $\mathbb{R}^d$ is

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strictly operator stable if there exists a linear operator B on $\mathbb{R}^d$ such that

$$\nu^t = t^B \nu \quad \text{for all } t > 0. \tag{2.1}$$

The linear operator B is called an *exponent* of ν. A stochastic process $\{X_t : t \geq 0\}$ on $\mathbb{R}^d$ is *proper* if for all $t > 0$ the distribution of X_t is full, and *continuous in law* if $X_{t_n} \Rightarrow X_t$ whenever $t_n \to t$, where "$\Rightarrow$" denotes convergence in distribution. A stochastic process $\{X_t : t \geq 0\}$ on $\mathbb{R}^d$ is *operator self-similar* if it is continuous in law and there exists a linear operator B on $\mathbb{R}^d$ and nonrandom vectors $\{d_t\}_{t\geq 0}$ in $\mathbb{R}^d$ such that

$$\{X_{ct}\} \stackrel{d}{=} \{c^B X_t + d_c\} \quad \text{for all } c > 0$$

where $\stackrel{d}{=}$ means the equality of all finite dimensional marginal distributions of the processes. The linear operator B is called an *exponent* of $\{X_t\}$. Hudson and Mason [7] show that if $\{X_t\}$ is a proper stochastic process on $\mathbb{R}^d$ that is continuous in law and has stationary independent increments with $X_0 = 0$, then $\{X_t\}$ is operator self-similar if and only if X_1 is strictly operator stable. In this case, every exponent B of X_1 is also an exponent of $\{X_t\}$. We will call this process an *operator Lévy motion*. Note that X_t has a bounded continuous density $f_t(x)$; see, e.g., Theorem 4.10.2. in [8]. If $B = aI$, a scalar multiple of the identity, then $\{X_t\}$ is a stable Lévy motion with index $\alpha = a^{-1}$. If $a = 1/2$ then $\{X_t\}$ is a Brownian motion.

The *Hausdorff dimension* of a set $E \subseteq \mathbb{R}^d$ is defined by

$$\dim(E) = \sup\{\beta : \Lambda^\beta(E) = \infty\} = \inf\{\beta : \Lambda^\beta(E) = 0\}$$

where $\Lambda^\beta(E) = \lim_{\varepsilon\downarrow 0} \Lambda^\beta_\varepsilon(E)$ and $\Lambda^\beta_\varepsilon(E) = \inf \sum_i \operatorname{diam}(E_i)^\beta$ where $\{E_i\}$ is a countable cover of E and the infimum is taken over all countable covers with $\sup_i \operatorname{diam}(E_i) \leq \varepsilon$.

Suppose that $\{X_t\}$ is an operator Lévy motion with exponent B. Factor the minimal polynomial of B into $f_1(x) \cdots f_p(x)$ where all roots of f_i have real part a_i and $a_i < a_j$ for $i < j$. Let $d_i = \dim \ker f_i(B)$ and $\alpha_i = a_i^{-1}$ so that $\alpha_1 > \cdots > \alpha_p$. Now we state our main results.

Theorem 2.1. *If $d_1 > 1$ or if $\alpha_1 \leq 1$ then $\dim X_{[0,1]} = \alpha_1$ almost surely.*

If $d = 1$ and $\alpha_1 = \alpha > 1$ then $\{X_t\}$ is an α-stable Lévy motion on $\mathbb{R}$ and it is known by Blumenthal and Getoor [5] that $\dim X_{[0,1]} = 1$ almost surely. Hence in case $d_1 = 1$ and $\alpha_1 > 1$ we do only have to consider $d \geq 2$, where we compute the Hausdorff dimension of the sample paths under an additional assumption in the following theorem.

Theorem 2.2. *If $\alpha_1 > 1$, $d_1 = 1$, and $d \geq 2$ then $\dim X_{[0,1]} = \rho = 1 + \alpha_2(1 - 1/\alpha_1)$ almost surely, as long as the density $f_t(x)$ of X_t is positive at $x = 0$.*

The remaining case where $\alpha_1 > 1$, $d_1 = 1$, $d \geq 2$ and $f_t(0) = 0$ for some (and hence for all) $t > 0$ seems to require some additional information about the support of operator stable densities. The corresponding result for stable Lévy motions hinges on the fact that the support of a stable density is a convex cone in $\mathbb{R}^d$, see Taylor [20]. However, the proof of this fact about stable densities in [20] was incorrect. The first correct proof appeared in Port and Vitale [14] and then a simpler geometric proof was provided by Ashbaugh, Rajput, Rama-Murthy, and Sundberg [1]. This result was extended to certain infinitely divisible laws in Rajput [17] but that class does not include the operator stable laws.

3. Proofs

Theorem 2 in Pruitt [15] shows that for any Lévy process $\{X_t\}$ we have $\dim X_{[0,1]} = \gamma$ almost surely where

$$\gamma = \sup\left\{\delta \geq 0 : \int_0^1 E\left(\|X_t\|^{-\delta}\right) dt < \infty\right\}. \tag{3.1}$$

The spectral decomposition for operator stable laws [9, 10, 11] shows that the tail behavior of the operator stable random vector X_t is determined by the real parts of the eigenvalues of the exponent B. Factor the minimal polynomial of B into $f_1(x)\cdots f_p(x)$ where all roots of f_i have real part a_i and $a_i < a_j$ for $i < j$. Define $V_i = \operatorname{Ker}(f_i(B))$. Then $V_1 \oplus \cdots \oplus V_p$ is a direct sum decomposition of $\mathbb{R}^d$ into B-invariant subspaces, and we may write $B = B_1 \oplus \cdots \oplus B_p$, where $B_i : V_i \to V_i$ and every eigenvalue of B_i has real part equal to a_i. The matrix for B in an appropriate basis is then block-diagonal with p blocks, the ith block corresponding to the matrix for B_i. Write $X_t = X_t^{(1)} + \cdots + X_t^{(p)}$ with respect to this direct sum decomposition. Since V_i is an B-invariant subspace it follows easily that $\{X_t^{(i)}\}$ is an operator Lévy motion on the d_i-dimensional vector space V_i with exponent B_i. Let $X = X_1$ and $X^{(i)} = X_1^{(i)}$ for $i = 1, \ldots, p$. Then (2.1) implies that $X_t \stackrel{d}{=} t^B X$ and $X_t^{(i)} \stackrel{d}{=} t^{B_i} X^{(i)}$ for all $t > 0$. Choose an inner product $\langle \cdot, \cdot \rangle$ on $\mathbb{R}^d$ such that $V_i \perp V_j$ for $i \neq j$, and let $\|x\|^2 = \langle x, x\rangle$ be the associated Euclidean norm. Then

$$\|t^B X\|^2 = \|t^{B_1} X^{(1)}\|^2 + \cdots + \|t^{B_p} X^{(p)}\|^2 \quad \text{for all } t > 0. \tag{3.2}$$

Now we begin the proof of Theorem 2.1. The following result is elementary but we could not find a suitable reference.

Lemma 3.1. *Let X be a random vector on $\mathbb{R}^d$ with continuous density $g(x)$.*

(a) *If $0 < \delta < d$ then $E(\|X\|^{-\delta}) < \infty$;*
(b) *If $\delta \geq d$ and $g(0) > 0$ then $E(\|X\|^{-\delta}) = \infty$.*

Proof. If $0 < \delta < d$ then

$$\begin{aligned} E(\|X\|^{-\delta}) &= \int_{x \in \mathbb{R}^d} \|x\|^{-\delta} g(x) dx \\ &= \int_{\|x\| \leq 1} \|x\|^{-\delta} g(x) dx + \int_{\|x\| > 1} \|x\|^{-\delta} g(x) dx \end{aligned}$$

$$\leq C \int_{\|x\| \leq 1} \|x\|^{-\delta} dx + 1$$

where $C = \max\{g(x) : \|x\| \leq 1\}$. Hence $E(\|X\|^{-\delta}) < \infty$ since $0 < \delta < d$. If $\delta \geq d$ and $g(0) > 0$ then for some $\varepsilon > 0$ we have $C_0 = \min\{g(x) : \|x\| \leq \varepsilon\} > 0$ and then

$$\begin{aligned} E(\|X\|^{-\delta}) &\geq \int_{\|x\| \leq \varepsilon} \|x\|^{-\delta} g(x) dx \\ &\geq C_0 \int_{\|x\| \leq \varepsilon} \|x\|^{-\delta} dx \end{aligned}$$

diverges since $\delta \geq d$. □

Lemma 3.2. *If $q > a_i$ and $\delta > 0$ then for some $C_i > 0$ we have*

$$\|t^{-B_i}\|^{\delta} \leq C_i t^{-q\delta} \quad \text{for all } 0 < t \leq 1. \tag{3.3}$$

Proof. Since $q > a_i$ and every eigenvalue of B_i has real part equal to a_i, it is easy to check (see, e.g., Corollary 2.2.5 in [11]) that $u^{-q}\|u^{B_i}\| \to 0$ as $u \to \infty$. Then for some $C > 0$ we have $u^{-q}\|u^{B_i}\| \leq C$ for all $u \geq 1$ and hence $\|u^{B_i}\| \leq Cu^q$ for all $u \geq 1$. Substituting $t = 1/u$ yields $\|t^{-B_i}\| \leq Ct^{-q}$ for all $0 < t \leq 1$ and (3.3) follows easily. □

Lemma 3.3. *In* (3.1) *we have* $\gamma \geq \min\{\alpha_1, d_1\}$.

Proof. Suppose that $\delta < \min\{\alpha_1, d_1\}$. Take $q \in (a_1, 1/\delta)$ noting that $\delta < \alpha_1 = 1/a_1$ so $1/\delta > a_1$ and the interval is nonempty. It follows from (3.2) that $\|t^B X\|^2 \geq \|t^{B_1} X^{(1)}\|^2$ so that $\|t^B X\|^{-\delta} \leq \|t^{B_1} X^{(1)}\|^{-\delta}$. Since $q > a_1$ Lemma 3.2 implies that (3.3) holds for $i = 1$. Then since $\|Ax\| \geq \|x\|/\|A^{-1}\|$ for any vector $x \in \mathbb{R}^d$ and any invertible linear operator A on $\mathbb{R}^d$, so that $\|Ax\|^{-\delta} \leq \|x\|^{-\delta}\|A^{-1}\|^{\delta}$, we have

$$\begin{aligned} I_\delta &= \int_0^1 E(\|X_t\|^{-\delta}) dt \\ &= \int_0^1 E(\|t^B X\|^{-\delta}) dt \\ &\leq \int_0^1 E(\|t^{B_1} X^{(1)}\|^{-\delta}) dt \\ &\leq \int_0^1 E(\|X^{(1)}\|^{-\delta} \|t^{-B_1}\|^{\delta}) dt \\ &\leq \int_0^1 E(\|X^{(1)}\|^{-\delta} C_1 t^{-q\delta}) dt \\ &= E(\|X^{(1)}\|^{-\delta}) \int_0^1 C_1 t^{-q\delta} dt \end{aligned}$$

where the last integral converges since $q\delta < 1$ and $E(\|X^{(1)}\|^{-\delta})$ exists in view of Lemma 3.1 (a). Note that $X^{(1)}$ has a continuous density since it is operator stable with exponent B_1. Then $I_\delta < \infty$ which implies that $\gamma > \delta$. □

Lemma 3.4. *If $q < a$ for every eigenvalue a of B and $\delta > 0$ then for some $C > 0$ we have*

$$\|t^B\|^{-\delta} \geq Ct^{-q\delta} \quad \text{for all } 0 < t \leq 1.$$

Proof. Since $q < a$ and every eigenvalue of B has real part greater than or equal to a, it is easy to check (see, e.g., Corollary 2.2.5 in [11]) that $u^q\|u^{-B}\| \to 0$ as $u \to \infty$. The rest of the argument is similar to Lemma 3.2. □

Lemma 3.5. *In* (3.1) *we have* $\gamma \leq \min\{\alpha_1, d\}$.

Proof. Suppose that $\delta > \min\{\alpha_1, d\}$. If $d < \alpha_1$ then $d = 1$ and $\alpha_1 > 1$, since $\alpha_i \leq 2$ in general. Then since $\delta > d = 1$ we have $E(\|X\|^{-\delta}) = \infty$ by Lemma 3.1 (b), because in this case $X = X^{(1)}$ is an α_1-stable random variable whose density is strictly positive; see, e.g., Remark 4 in §2.2 of [23]. Since $\|Ax\| \leq \|A\| \cdot \|x\|$ we also have $\|Ax\|^{-\delta} \geq \|A\|^{-\delta}\|x\|^{-\delta}$ for any vector $x \in \mathbb{R}^d$ and any nonzero linear operator A on $\mathbb{R}^d$, so that

$$\begin{aligned}
I_\delta &= \int_0^1 E(\|X_t\|^{-\delta})dt \\
&= \int_0^1 E(\|t^B X\|^{-\delta})dt \\
&\geq \int_0^1 E(\|t^B\|^{-\delta}\|X\|^{-\delta})dt \\
&= E(\|X\|^{-\delta}) \int_0^1 \|t^B\|^{-\delta} dt = \infty
\end{aligned}$$

in this case. On the other hand, if $d \geq \alpha_1$ then $\delta > \alpha_1$. Choose $q \in (1/\delta, a_1)$ so that $q\delta > 1$. Then Lemma 3.4 implies

$$\begin{aligned}
I_\delta &= \int_0^1 E(\|X_t\|^{-\delta})dt \\
&\geq E(\|X\|^{-\delta}) \int_0^1 Ct^{-q\delta} dt = \infty
\end{aligned}$$

in this case as well. Hence in either case we have shown that $\gamma < \delta$. □

Proof of Theorem 2.1. Lemmas 3.3 and 3.5 show that

$$\min\{\alpha_1, d_1\} \leq \gamma \leq \min\{\alpha_1, d\}.$$

If $d_1 \geq 2$ then this shows that $\gamma = \alpha_1$, since $\alpha_i \leq 2$ in general. If $\alpha_1 \leq 1$ then the same is true, since d_1 is a positive integer. If $d = 1$, hence $d_1 = 1$, and $\alpha_1 > 1$ then $\gamma = 1$, so that we recover the result of Blumenthal and Getoor [5] in this case. □

Now we turn to the remaining case where $\alpha_1 > 1$, $d_1 = 1$, and $d \geq 2$. We want to show that $\dim X_{[0,1]} = \rho = 1 + \alpha_2(1 - 1/\alpha_1)$. It is easy to check that $\rho \in (\alpha_2, \alpha_1)$ and also that $\rho > 1$. Lemmas 3.3 and 3.5 only yield that $1 \leq \gamma \leq \alpha_1$ so we must turn to different methods. Theorem 1 in Pruitt [15] shows that for any Lévy process $\{X_t\}$ we have $\dim X_{[0,1]} = \gamma$ almost surely where

$$\gamma = \sup\left\{\delta \geq 0 : \limsup_{b\to 0} b^{-\delta} \int_0^1 P\left(\|X_t\| \leq b\right) dt < \infty\right\}. \tag{3.4}$$

Lemma 3.6. *If $\alpha_1 > 1$, $d_1 = 1$, and $d \geq 2$ then in (3.4) we have $\gamma \geq \rho$.*

Proof. Using the norm and the spectral decomposition of X_t and B from the beginning of this section, we have

$$\begin{aligned} P(\|X_t\| \leq b) &= P(\|X_t\|^2 \leq b^2) \\ &\leq P(|X_t^{(1)}|^2 + \|X_t^{(2)}\|^2 \leq b^2) \\ &\leq P(|X_t^{(1)}| \leq b, \|X_t^{(2)}\| \leq b) \\ &\leq P(|X^{(1)}| \leq t^{-1/\alpha_1} b, \|X^{(2)}\| \leq \|t^{-B_2}\| b), \end{aligned}$$

using the fact that $X^{(1)}$ is a stable random variable with index α_1, and the inequality $\|Ax\| \geq \|x\|/\|A^{-1}\|$ for any vector $x \in \mathbb{R}^d$ and any invertible linear operator A on $\mathbb{R}^d$. Since every eigenvalue of B_2 has real part equal to $a_2 = 1/\alpha_2$, Lemma 3.2 with $\delta = 1$ implies that for any $\varepsilon > 0$ there exists a $C > 0$ such that

$$\|t^{-B_2}\| \leq C t^{-(a_2+\varepsilon)} \quad \text{for all } 0 < t \leq 1.$$

Let $g(x^{(1)}, x^{(2)})$ be the joint density of the operator stable random vector $(X^{(1)}, X^{(2)})$ and let

$$M = \max\{g(x^{(1)}, x^{(2)}) : (x^{(1)}, x^{(2)}) \in V_1 \oplus V_2\}.$$

Since any operator stable random vector has a bounded continuous density (see, e.g., Jurek and Mason [8] Theorem 4.10.2) we have $0 < M < \infty$. Then there exists a constant $\widetilde{C} > 0$ depending only on the norm $\|.\|$ on V_2 such that

$$\begin{aligned} P(\|X_t\| \leq b) &\leq P(|X^{(1)}| \leq t^{-1/\alpha_1} b, \|X^{(2)}\| \leq C t^{-(a_2+\varepsilon)} b) \\ &= \int\int I(|x^{(1)}| \leq t^{-1/\alpha_1} b, \|x^{(2)}\| \leq C t^{-(a_2+\varepsilon)} b) g(x^{(1)}, x^{(2)}) dx^{(1)} dx^{(2)} \\ &\leq M(2t^{-1/\alpha_1} b)(C t^{-(a_2+\varepsilon)} b)^{d_2} \\ &= M_1 b^{1+d_2} t^{-1/\alpha_1 - d_2/\alpha_2 - \varepsilon d_2} \end{aligned}$$

where $M_1 > 0$ is some constant not depending on b and t. Similarly, we also have

$$\begin{aligned} P(\|X_t\| \leq b) &\leq P(|X_t^{(1)}| \leq b) \\ &= P(|X^{(1)}| \leq t^{-1/\alpha_1} b) \\ &\leq M(2t^{-1/\alpha_1} b) \end{aligned}$$

and so for $0 < b < 1$

$$\begin{aligned}\int_0^1 P(\|X_t\| \le b)dt &\le \int_0^{b^{\alpha_2}} P(\|X_t\| \le b)dt + \int_{b^{\alpha_2}}^{\infty} P(\|X_t\| \le b)dt \\ &\le 2Mb \int_0^{b^{\alpha_2}} t^{-1/\alpha_1} dt + M_1 b^{1+d_2} \int_{b^{\alpha_2}}^{\infty} t^{-1/\alpha_1 - d_2/\alpha_2 - \varepsilon d_2} dt \\ &= M_2 b^{\rho} + M_3 b^{\rho - \varepsilon_1}\end{aligned}$$

where $M_2 = 2M/(1 - 1/\alpha_1) > 0$, $\varepsilon_1 = \varepsilon d_2 \alpha_2$, and

$$M_3 = \frac{-M_1}{1 - 1/\alpha_1 - d_2/\alpha_2 - \varepsilon d_2} > 0$$

since $1/\alpha_1 + d_2/\alpha_2 > 2/\alpha_1 \ge 1$. Given $\delta > 0$ with $\delta < \rho$ choose $\varepsilon_1 > 0$ small enough so that $\rho - \varepsilon_1 > \delta$ (i.e., choose $\varepsilon = \varepsilon_1/(d_2\alpha_2)$ small enough). Then

$$b^{-\delta} \int_0^1 P(\|X_t\| \le b)dt \le b^{-\delta}(M_3 b^{\rho} + M_4 b^{\rho - \varepsilon_1}) \to 0 \quad \text{as } b \to 0, \tag{3.5}$$

since $\rho - \varepsilon_1 - \delta > 0$. Since $\delta < \rho$ is arbitrary this shows that $\gamma \ge \rho$. □

Lemma 3.7. *If $\alpha_1 > 1$, $d_1 = 1$, and $d \ge 2$ then in* (3.4) *we have $\gamma \le \rho$, as long as the density $g(x) = f_1(x)$ of X_1 is positive at $x = 0$.*

Proof. Since X_1 is operator stable it has a bounded continuous density $g(x_1, \dots, x_d)$ (see, e.g., [8] Theorem 4.10.2) and hence for some $m > 0$ and $L > 0$ we have $g(x_1, \dots, x_d) \ge L$ whenever $|x_i| \le m$ for all $i = 1, \dots, d$. Now choose b_1 such that

$$b^{\alpha_1} m^{-\alpha_1} p^{-\alpha_1/2} < 1 \quad \text{for all } 0 < b < b_1$$

where p is the number of spectral components as defined in the beginning of this section. Choose $\varepsilon > 0$ small enough to make $a_2 - \varepsilon > a_1$ and use Lemma 3.4 with $\delta = 1$ to obtain a constant $C > 0$ such that

$$\|t^{B_i}\| \le C t^{a_2 - \varepsilon} \quad \text{for all } 0 < t \le 1 \text{ and all } i = 2, \dots, p. \tag{3.6}$$

Then choose $\delta > 0$ with $1/(a_2 - \varepsilon) < \delta < 1/a_1$ and note that $1 + \delta(\varepsilon - a_2) < 0$. Next choose b_2 such that

$$p^{-1/2} C^{-1} b^{1+\delta(\varepsilon - a_2)} \ge m \quad \text{for all } 0 < b < b_2. \tag{3.7}$$

Note that $\delta < 1/a_1 = \alpha_1$ so that $b^{\alpha_1}/b^{\delta} \to 0$ as $b \to 0$. Now choose b_3 such that

$$1 > b^{\delta} > b^{\alpha_1} m^{-\alpha_1} p^{-\alpha_1/2} \quad \text{for all } 0 < b < b_3 \tag{3.8}$$

and finally take $b_0 = \min\{1, b_1, b_2, b_3\}$. Then for all $0 < b < b_0$ we have

$$\begin{aligned}
P(\|X_t\| \le b) &= P(\|X_t^{(1)}\|^2 + \cdots + \|X_t^{(p)}\|^2 \le b^2)\\
&\ge P(\|X_t^{(i)}\|^2 \le b^2/p \;\forall\; i = 1, \ldots, p)\\
&= P(\|t^{B_i} X^{(i)}\| \le bp^{-1/2} \;\forall\; i = 1, \ldots, p)\\
&\ge P(|X^{(1)}| \le bp^{-1/2} t^{-1/\alpha_1}, \|X^{(i)}\| \le bp^{-1/2}\|t^{B_i}\|^{-1} \;\forall\; i = 2, \ldots, p)
\end{aligned}$$

so that for $0 < b < b_0$

$$\begin{aligned}
I_b &= \int_0^1 P(\|X_t\| \le b)dt\\
&\ge \int_{b^{\alpha_1} m^{-\alpha_1} p^{-\alpha_1/2}}^{b^\delta} P(|X^{(1)}| \le bp^{-1/2} t^{-1/\alpha_1}, \|X^{(i)}\| \le bp^{-1/2}\|t^{B_i}\|^{-1} \;\forall\; i = 2, \ldots, p)dt
\end{aligned}$$

where (3.8) ensures that the domain of integration is contained in $(0, 1)$. Since (3.6) and (3.7) imply that

$$bp^{-1/2}\|t^{B_i}\|^{-1} \ge bp^{-1/2} C^{-1} t^{\varepsilon - a_2} \ge bp^{-1/2} C^{-1} (b^\delta)^{\varepsilon - a_2} = p^{-1/2} C^{-1} b^{1+\delta(\varepsilon - a_2)} \ge m$$

we also have

$$I_b \ge \int_{b^{\alpha_1} m^{-\alpha_1} p^{-\alpha_1/2}}^{b^\delta} P(|X^{(1)}| \le bp^{-1/2} t^{-1/\alpha_1}, \|X^{(i)}\| \le m \;\forall\; i = 2, \ldots, p)dt$$

and since in this integral

$$bp^{-1/2} t^{-1/\alpha_1} \le bp^{-1/2} (b^{\alpha_1} m^{-\alpha_1} p^{-\alpha_1/2})^{-1/\alpha_1} = bp^{-1/2} b^{-1} m p^{1/2} = m$$

we also have that

$$\begin{aligned}
I_b &\ge \int_{b^{\alpha_1} m^{-\alpha_1} p^{-\alpha_1/2}}^{b^\delta} \int I(|x^{(1)}| \le bp^{-1/2} t^{-1/\alpha_1}, \|x^{(i)}\| \le m \;\forall\; i = 2, \ldots, p) g(x)dxdt\\
&\ge \int_{b^{\alpha_1} m^{-\alpha_1} p^{-\alpha_1/2}}^{b^\delta} L(2bp^{-1/2} t^{-1/\alpha_1}) \prod_{i=2}^p (2m/d_i)^{d_i} dt
\end{aligned}$$

using the fact that $\{|x_j| \le r/d_i \;\forall j = 1, \ldots, d_i\} \subseteq \{\|x\| \le r\}$ on any d_i dimensional vector space. Then

$$\begin{aligned}
I_b &\ge C_0 b \int_{b^{\alpha_1} m^{-\alpha_1} p^{-\alpha_1/2}}^{b^\delta} t^{-1/\alpha_1} dt\\
&= C_1 b^{1+\delta(1-1/\alpha_1)} - C_2 b^{\alpha_1}
\end{aligned}$$

where C_0, C_1, C_2 are positive real constants independent of b. Since $\delta = \alpha_2 + \varepsilon_0$ for some $\varepsilon_0 > 0$ arbitrarily small if $\varepsilon > 0$ is small enough and $\delta > 1/(a_2 - \varepsilon)$ is chosen close to $1/(a_2 - \varepsilon)$, and since $\rho < \alpha_1$, we may assume that

$$1 + \delta(1 - 1/\alpha_1) = 1 + \alpha_2(1 - 1/\alpha_1) + \varepsilon_0(1 - 1/\alpha_1) = \rho + \varepsilon_0(1 - 1/\alpha_1) < \alpha_1.$$

Then $b^{\alpha_1}/b^{1+\delta(1-1/\alpha_1)} \to 0$ as $b \to 0$, so that

$$I_b = \int_0^1 P(\|X_t\| \le b)dt \ge (C_1/2)b^{1+\delta(1-1/\alpha_1)}$$

for all $b > 0$ sufficiently small. Now given $\eta > 0$ with $\eta > \rho$, for any $\delta > 0$ such that

$$1 + \delta(1 - 1/\alpha_1) = \rho + \varepsilon_0(1 - 1/\alpha_1) < \eta$$

we have

$$b^{-\eta} \int_0^1 P(\|X_t\| \le b)dt \ge b^{-\eta}(C_1/2)b^{1+\delta(1-1/\alpha_1)} \to \infty \quad \text{as } b \to 0, \tag{3.9}$$

which shows that $\gamma \le \rho$. □

Proof of Theorem 2.2. Since $X_t \stackrel{d}{=} t^B X_1$ we have $g(0) = 0$ if and only if $f_t(0) = 0$ for all $t > 0$. Then the result follows immediately from Lemmas 3.6 and 3.7. □

Remark 3.8. If X_t has d independent stable marginals with index $\alpha_1 \ge \cdots \ge \alpha_d$ then a result of Pruitt and Taylor [16] implies that $\dim X_{[0,1]} = \rho = 1+\alpha_2(1-1/\alpha_1)$ almost surely whenever $\alpha_1 > 1$ and $d \ge 2$. We conjecture that the $\dim X_{[0,1]} = \rho$ almost surely for any operator Lévy motion with $\alpha_1 > 1$, $d_1 = 1$, and $d \ge 2$, but we have only been able to prove this in the case where the density $f_t(x)$ of X_t is strictly positive at $x = 0$. If $f_t(0) = 0$, the argument depends on showing that the density $f_1(x) > 0$ on a suitable set of $x \in \mathbb{R}^d$ near the origin. For example, if $d = 2$ and $X_t^{(2)}$ is a stable subordinator with Laplace transform $\exp(-tCs^{-\alpha_2})$ then the density of X_t is strictly positive on the half-space $\{x_2 > 0\}$. Then the argument of Lemma 3.7 can be extended using a positive lower bound L on the set $-m \le x_1 \le m$, $m/2 \le x_2 \le m$. In order to make this argument general, we require a characterization of the support of operator stable densities similar to what is known for stable densities. Since $X_t \stackrel{d}{=} t^B X_1$ the scaling relation is $f_t(x) = f_1(t^{-B}x)t^{-\operatorname{trace}(B)}$ and since the orbits $\{t^{-B}x : t > 0\}$ are not straight lines, the geometrical situation is more complicated in this case.

Remark 3.9. There are several interesting open problems concerning operator stable sample paths. Taylor [20] shows that the asymptotic behavior of the first passage times, occupation times and maxima of a d-dimensional Lévy motion are different for processes of type A (density is positive at $x = 0$) or type B (density is zero at $x = 0$). It would be interesting to extend these results to operator Lévy motions. Pruitt and Taylor [16] compute the exact Hausdorff dimension for the sample paths of an operator stable Lévy motion with independent stable marginals, using the density theorem of Rogers and Taylor [18] together with sharp bounds on the asymptotics of occupation times. Extending their result to operator Lévy motions would sharpen the results of this paper. The index γ', defined by (3.4) with lim inf instead of lim sup, was first considered by Hendricks [6]. Taylor [21] shows that sample paths of a general Lévy process have packing dimension γ' almost surely. The excellent review article of Taylor [19] defines a *random fractal* as a random set whose packing and Hausdorff dimensions are equal to each other, and greater than the topological dimension. Sample paths of stable Lévy motions are random fractals. Under the conditions of Theorem 2.2, the index γ' can be used to show that the packing dimension and Hausdorff dimension of the random set $X_{[0,1]}$ are both equal to $\rho = 1+\alpha_2(1-1/\alpha_1) > 1$ almost surely. Hence $X_{[0,1]}$ is a random fractal in this case. It is not known whether the sample paths of all operator Lévy motions are random fractals. It would also be interesting to compute the (exact) Hausdorff dimension of the random set $X_E = \{X_t(\omega) : t \in E\}$ when E is a Borel subset of $[0,1]$ with Hausdorff dimension η. The classical methods of Blumenthal and Getoor [5] together with estimates of the present paper can be used to show that $\alpha_p \eta \leq \dim X_E \leq \alpha_1 \eta$ almost surely, for an operator Lévy motion on $\mathbb{R}^d$ with $d \geq 2$. Details are available from the authors upon request. If X_t is spectrally simple (every eigenvalue of the exponent B has the same real part $1/\alpha$) then this shows that $\dim X_E = \alpha\eta$ almost surely. It does not seem possible to generalize this result to arbitrary operator Lévy motions without using the more sophisticated methods of Pruitt and Taylor [16]. In porous media flow, (operator) Lévy motions model anomalous diffusion, in which a contaminant plume spreads faster than the classical Brownian motion model predicts [2, 3, 4, 12]. The anomalous spreading rate is related to fractal properties of the porous medium [22]. The relation between the sample path properties of particle traces and the geometry of the porous medium is not yet understood.

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