On the distribution of Continuous Time Random Walk limits

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1. Density of the limit distribution

In the talk by Peter Becker-Kern it was shown that the limit distribution of the CTRW $X(t)$ is given by

$$\begin{aligned}\lim_{c\to\infty} P\{\tilde{B}(c)X(ct)\in M\} &= P\{A(E(t))\in M\}\\ &= -\int_0^\infty L_D(H_u)(t)du\end{aligned}$$

where

$$\begin{aligned}H_u(t) &= P\{A(u)\in M, D(u)\le t\}\\ &= \int_M\int_0^t f_u(y,r)\,dr\,dy\end{aligned}$$

$f_u(y,r)$ is the joint density of $(A(u),D(u))$ and L_D is the generator of the stable subordinator.

We have for the Laplace transform $\mathcal{L}(g)(s)=\int_0^\infty g(t)e^{-st}dt$ that

$$-\mathcal{L}(L_D(H_u))(s) = s^\beta\mathcal{L}(H_u)(s) = \mathcal{L}(\frac{\partial^\beta}{\partial t^\beta}H_u)(s)$$

where

$$\frac{\partial^\beta}{\partial t^\beta}g(t) = C_\beta\int_0^t \frac{\partial}{\partial t}g(t-\tau)\tau^{-\beta}\,d\tau$$

is a *fractional derivative* of order $0<\beta<1$.

Hence

$$P\{A(E(t)) \in M\} = \int_0^\infty \frac{\partial^\beta}{\partial t^\beta} H_u(t)\, du$$

Theorem 1 *The density of $A(E(t))$ is given by*

$$h(x,t) = \int_0^\infty \frac{\partial^{\beta-1}}{\partial t^{\beta-1}} f_u(x,t) du$$

where

$$\frac{\partial^{\beta-1}}{\partial t^{\beta-1}} f_u(x,t) = \tilde{C}_\beta \int_0^t f_u(x, t-\tau)\tau^{-\beta}\, d\tau$$

is a fractional integral of order $1-\beta$ in the time variable.

Note that the fractional integral is a non-local operator and given by convolution with the weakly singular function $\tau^{-\beta}$.

2. Fourier-Laplace-Transform

For suitable functions $g(x,t)$ on $\mathbb{R}^d \times \mathbb{R}_+$ we define the *Fourier-Laplace-Transform* (FLT) by

$$\mathcal{FL}(g)(k,s) = \int_{\mathbb{R}^d} \int_0^\infty g(x,t) e^{i\langle k,x\rangle} e^{-st} dt\, dx$$

Since f_u is the density of the infinitely divisible distribution of $(A(u), D(u))$ we have by the Lévy-Khinchin representation

$$\mathcal{FL}(f_u)(k,s) = \exp(-u\psi(k,s))$$

and we call $\psi(k,s)$ the log-Fourier-Laplace-Transform of (A,D).

The FLT of the density $h(x,t)$ of the CTRW-limit $A(E(t))$ is given by

$$\begin{aligned} \mathcal{FL}(h)(k,s) &= s^{\beta-1} \int_0^\infty \mathcal{FL}(f_u)(k,s)\, du \\ &= s^{\beta-1} \int_0^\infty \exp(-u\psi(k,s))\, du \\ &= \frac{s^{\beta-1}}{\psi(k,s)} \end{aligned}$$

Therefore

Theorem 2

$$\mathcal{FL}(h)(k,s) = \frac{s^{\beta-1}}{\psi(k,s)} \quad , (k,s) \in \mathbb{R}^d \times \mathbb{R}_+.$$

Equivalently, we can write

$$\psi(k,s)\mathcal{F}\mathcal{L}(h)(k,s) = s^{\beta-1}. \tag{1}$$

If we define the ψdo-differential operator $\psi(iD_x, \partial_t)$ by

$$\psi(iD_x, \partial_t)g(x,t) = \mathcal{F}\mathcal{L}^{-1}(\psi(k,s)\mathcal{F}\mathcal{L}(g)(k,s))(x,t)$$

and since

$$\mathcal{F}\mathcal{L}\big(\delta(x)\frac{t^{-\beta}}{\Gamma(1-\beta)}\big)(k,s) = s^{\beta-1}$$

we can rewrite (1) as the ψdo-differential equation

$$\psi(iD_x, \partial_t)h(x,t) = \delta(x)\frac{t^{-\beta}}{\Gamma(1-\beta)}. \tag{2}$$

This is a generalization of the fractional kinetic equation

$$\frac{\partial^\beta}{\partial t^\beta}h(x,t) = \frac{\partial^2}{\partial x^2}h(x,t) + \delta(x)\frac{t^{-\beta}}{\Gamma(1-\beta)} \tag{3}$$

used to model anomalous diffusion.

(2) reduces to (3) if the waiting times J_i are independent of the jumps Y_i and Y_i is attracted to a normal random variable A, since in this case we have by independence of A and D that

$$\psi(k,s) = k^2 + s^\beta.$$

3. Examples

(A) Let D be a stable subordinator with Laplace transform $\mathbb{E}(e^{-sD}) = \exp(-s^\beta)$, $0 < \beta < 1$ and define the conditional distribution

$$A|D = t \sim \mathcal{N}(0, 2t).$$

Then

$$\mathbb{E}(e^{ikA}) = \mathbb{E}[\mathbb{E}[e^{ikA}|D]] = \mathbb{E}[e^{-k^2 D}] = \exp(-|k|^{2\beta})$$

so A is symmetric stable with index 2β.

The log-FLT of (A, D) is given by

$$\psi(k, s) = (k^2 + s)^\beta.$$

Now let (Y_i, J_i) be i.i.d. as (A, D). By Theorem 2 the FLT of the density h of the CTRW limit $A(E(t))$ is given by

$$\mathcal{F}\mathcal{L}(h)(k, s) = \frac{s^{\beta-1}}{(k^2 + s)^\beta}.$$

Inverting the Laplace-transform first yields

$$\mathcal{F}(h)(k, t) = \int_0^t e^{-k^2 u} \frac{u^{\beta-1}}{\Gamma(\beta)} \frac{(t-u)^{-\beta}}{\Gamma(1-\beta)} du.$$

Inverting the Fourier-transform yields

$$h(x,t) = \int_0^t \frac{1}{\sqrt{4\pi u}} \exp\left(-\frac{x^2}{4u}\right) \frac{u^{\beta-1}}{\Gamma(\beta)} \frac{(t-u)^{-\beta}}{\Gamma(1-\beta)} du.$$

Hence

$$A(E(t)) \overset{d}{=} (tB)^{1/2} Z, \tag{4}$$

where $Z \sim \mathcal{N}(0,2)$ and B has a Beta-distribution independent of Z. From (4) we get the scaling relation

$$A(E(t)) \overset{d}{=} t^{1/2} A(E(1))$$

so $A(E(t))$ models a diffusion.

The ψdo differential equation for h is given by (2)

$$\left(\frac{\partial}{\partial t} - \frac{\partial^2}{\partial x^2}\right)^\beta h(x,t) = \delta(x) \frac{t^{-\beta}}{\Gamma(1-\beta)}.$$

(B) Now define the conditional distribution

$$A|D = t \sim \mathcal{N}(0, 2t^m)$$

for some $m > \beta$. Then the CTRW limit $M(t) = A(E(t))$ fulfills the scaling relation

$$M(ct) \overset{d}{=} c^{m/2} M(t), \quad c,t > 0.$$

Therefore we have the following cases

- $\beta < m < 1$: subdiffusive;
- $m = 1$: diffusive;
- $m > 1$: superdiffusive.

(C) As before let D be a stable subordinator. Take U independent of D with

$$P\{U = \pm 1\} = \frac{1}{2}$$

and let

$$Y = UD^m, \quad m > \frac{\beta}{2}.$$

Take (Y_i, J_i) i.i.d. as (Y, D). Then the CTRW limit $M(t) = A(E(t))$ has the scaling relation

$$M(ct) = c^m M(t), \quad c, t > 0.$$

Therefore

- $\beta/2 < m < 1/2$: subdiffusive;
- $m = 1/2$: diffusive;
- $m > 1/2$: superdiffusive.