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RESEARCH ARTICLE

Mehler semigroups, Ornstein-Uhlenbeck processes and background driving Lévy processes on locally compact groups and on hypergroups

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Abstract For finite dimensional vector spaces it is well-known that there exists a 1-1-correspondence between distributions of Ornstein-Uhlenbeck type processes (w.r.t. a fixed group of automorphisms) and (background driving) Lévy processes, hence between M- or skew convolution semigroups on the one hand and continuous convolution semigroups on the other. An analogous result could be proved for simply connected nilpotent Lie groups. Here we extend this correspondence to a class of commutative hypergroups.

1 Introduction

Let $\mathbb{V}$ be a d-dimensional real vector space and let $(T_t)_{t\in\mathbb{R}}$ be a continuous one-parameter group of automorphisms. *M-semigroups* (*or skew convolution semigroups*) are continuous one-parameter families of probabilities $(\mu(t))_{t\geq 0}$ on $\mathbb{V}$ satisfying the *cocycle equation* $\mu(t+s)=\mu(t)\star T_t(\mu(s))$, $\forall s,t\geq 0$. These skew or M-semigroups are marginal distributions of (generalized) Ornstein-Uhlenbeck-processes (the corresponding semigroups of transition kernels are called *Mehler semigroups*) and correspond in a 1-1-manner to continuous convolution semigroups, the distributions of Lévy processes (called *background driving Lévy processes*). The correspondence is expressed by path-wise random integral representations of the involved processes. See [32] (for $d=1$), [2] or [43] and the literature mentioned there. More generally, for random integrals of additive processes see [49]. It should be mentioned that limits of M-semigroups (for $t\to\infty$) are self-decomposable laws and vice versa [50]. For the background of self-decomposability and random integral representations on

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vector spaces see e.g., the monograph [37], or [33, 38, 52], furthermore, [1, 49, 50], and the literature mentioned there. See also [35, 36]. For some applications of self-decomposability see e.g., [4, 34] and the references there.

In the past, the theory of M-(Mehler-)semigroups was also successfully investigated for non-locally compact convolution structures, e.g., infinite dimensional vector spaces and random measures resp. point processes. The reader may consult e.g., [5, 10, 41, 51] and the literature mentioned there.

In the above definitions of M-semigroups only the operator *semi*group $(T_t : t \geq 0, T_0 = I)$ is involved. But on finite dimensional vector spaces, as well as in the main examples in the sequel, Lie groups resp. matrix cone hypergroups, such a semigroup is uniquely extended to a group $(T_t)_{t\in\mathbb{R}}$ of operators resp. of automorphisms. Hence, in contrast to the infinite dimensional case, the restriction to continuous one-parameter groups $(T_t)_{t\in\mathbb{R}}$ is well motivated.

For locally compact groups $\mathbb{G}$ admitting a continuous one-parameter group $(T_t)_{t\in\mathbb{R}} \subseteq \mathrm{Aut}(\mathbb{G})$, Ornstein-Uhlenbeck processes (or Mehler semigroups of transition kernels) resp. M-semigroups of probabilities $(\mu(t))_{t\geq 0}$ on the one side, and Lévy processes resp. continuous convolution semigroups $(\mu_t)_{t\geq 0}$ on the other, are defined verbatim as in the vector space case. In the group case—as random integral representations are in general not available—at least for contractible simply connected nilpotent Lie groups a 1-1-correspondence between M-semigroups and continuous convolution semigroups is established via ‘forward resp. backward’ Lie-Trotter product formulas $(LT1)$ resp. $(LT2)$:

$$\mu(t) = \lim_{n\to\infty} \mathop{\bigstar}_{k=0}^{n-1} T_{\frac{kt}{n}}(\mu_{t/n}) \quad \text{resp. } \mu_t = \lim_{n\to\infty} \mu(t/n)^n \tag{1.1}$$

This may be understood as weak versions of random integral representations. See e.g., [21, §2.14], [19, Theorem C], [18]. (For a process-approach under some technical conditions see e.g., [40].)

The proof relies (i) on the construction of (space-time) Lévy processes resp. continuous convolution semigroups on the *space-time building* $\Gamma := \mathbb{G} \rtimes \mathbb{R}$, (ii) on the existence of common cores for generators of continuous convolution semigroups on Γ and (iii) on Lie-Trotter formulas for addition of generators of C_0-contraction semigroups resp. continuous convolution semigroups. The second property, the existence of common cores—proved independently and nearly simultaneously by J. Faraut, K. Harzallah, F. Hirsch, J.P. Roth, M. Duflo [14, 15, 25, 26, 28, 29, 47]—is crucial. See also [11, 12, 17, 23, 27, 31, 46]. (In fact, for our purpose slight generalizations of this result are needed, see Theorem 2.9 (c), (d), below.) The idea to relate M-semigroups on the one hand and space-time convolution semigroups on semi-direct extensions on the other is due to K.H. Hofmann and Z. Jurek [30] (with a slightly different approach).

As a corollary it follows that the Schwartz-Bruhat test functions $\mathcal{D}(\mathbb{G})$ and—for direct and semi-direct extensions $\Gamma = \mathbb{G} \rtimes \mathbb{R}$—also the subspaces $\mathcal{D}(G) \otimes \mathcal{D}(\mathbb{R}) \subseteq \mathcal{D}(\Gamma)$ are common cores for generators of continuous convolution semigroups on $\mathbb{G}$ and Γ respectively; a key result for the verification of $(LT1)$ and $(LT2)$. (Recall that for Lie groups $\mathcal{D}(\mathbb{G})$ is just $C_c^\infty(\mathbb{G})$.)

Recently M. Rösler [48] and M. Voit [53] investigated commutative hypergroup structures $(\mathcal{K}, \star)$ on the cone of non-negative definite $d \times d$-matrices with a *group like* behavior. (For convenience, here we restrict to the case of real matrices.) In particular, the structure of the automorphism group is well-known: $\mathrm{Aut}(\mathcal{K})$ is a homomorphic image of $\mathrm{GL}(\mathbb{R}^d)$. For $d = 1$ these hypergroups are just the well-known Bessel-Kingman hypergroups. In [20] some probabilistic aspects of these hypergroup structures were investigated, especially divisibility, (semi-)stability and also self-decomposability and M-semigroups. However, the problem of existence of background driving Lévy processes and the correspondence by Lie-Trotter formulas was not investigated there. This is the main target of the present investigations.

Note that a version of the above-mentioned theorem of F. Hirsch et al. for hypergroups is proved in the thesis S. Menges [44], 5.26. There also the existence of a common core for convolution semigroups on commutative hypergroups is established ([44], 5.17, 5.22). However, for non-Abelian hypergroups there is no natural candidate for a common core as e.g., $\mathcal{D}(\mathbb{G})$ for general locally compact groups. To find suitable function spaces on semi-direct extensions and to show a core property which allows to prove the analogues of $(LT1)$ and $(LT2)$ is a crucial tool of this investigation.

In Sect. 2 we collect notations and basic facts for continuous convolution semigroups of probabilities on groups and hypergroups and invariant C_0-contraction semigroups, including a proof of the afore mentioned Theorem of F. Hirsch et al. (in its slightly generalized form). In Sect. 3 we apply these results to the case of semi-direct extensions $\mathbb{G} \rtimes \mathbb{R}$ of locally compact groups (generalizing slightly the already known results for nilpotent Lie groups). Section 4 contains the main results: Theorems 4.1 and 4.2: The 1-1-correspondence between M-semigroups $(\mu(t))$ and (background driving) Lévy processes (μ_t) on groups and on hypergroups respectively. The proof of the first is a consequence of the results collected in Sect. 3, whereas Sect. 5 is concerned with the proof of Theorem 4.2, the hypergroup case: For a class of hypergroups containing the afore mentioned matrix cone hypergroups the existence of background driving Lévy processes and the correspondence via the Lie-Trotter formulas is established. The proof is quite technical and sometimes cumbersome, but I was unable to find a more elegant version.

In the appendix we discuss briefly an alternative approach to the mapping $(\mu_t)_{t \geq 0} \mapsto (\mu(t))_{t \geq 0}$ in the case of Abelian hypergroups.

2 Notations and basic facts

Let $\mathbb{G}$ be a locally compact group or a hypergroup. We adopt the following notations: $\mathcal{M}^1(\mathbb{G})$ denotes the set of probability measures, $\mathcal{M}^{(1)}(\mathbb{G}) := \{\lambda \in \mathcal{M}^b(\mathbb{G}) : \|\lambda\| \leq 1\}$, $\mathcal{M}^b(\mathbb{G})$ the set of bounded measures, $C_0(\mathbb{G})$ the Banach space of continuous functions vanishing at ∞. Point measures are denoted by $\varepsilon_x, x \in \mathbb{G}$. $\mathcal{D}(\mathbb{G})$ denotes the Schwartz-Bruhat test-function space and $\mathfrak{E}(\mathbb{G}) := \{g \in C_b(\mathbb{G}) : f \cdot g \in \mathcal{D}(\mathbb{G}) \ \forall f \in \mathcal{D}(\mathbb{G})\}$, the space of bounded regular functions. Note that for Lie groups we have $\mathcal{D}(\mathbb{G}) = C_c^\infty(\mathbb{G})$ and $\mathfrak{E}(\mathbb{G}) = C_b^\infty(\mathbb{G})$. For details see e.g., [21, 22]. If not otherwise stated, measure spaces are always endowed with the topology of weak convergence and convergence of operators is understood as convergence in the strong operator topology.

According to the Riesz representation theorem, measures $\mu \in \mathcal{M}^b(\mathbb{G})$ are identified with continuous linear functionals on $C_0(\mathbb{G})$, the dual pairing is denoted by $\int_{\mathbb{G}} f\,\mathrm{d}\mu = \langle f, \mu\rangle$. Measures are also identified with linear operators R_μ, L_μ, the convolution operators acting on $C_0(\mathbb{G})$ from right resp. left. (Cf. the operators T_{μ^-}, T^{μ^-} in [6].)

$$R_\mu: \quad (R_\mu f)(x) := \int f\,\mathrm{d}(\varepsilon_x \star \mu) = \langle f, \varepsilon_x \star \mu\rangle,$$

$$L_\mu: \quad (L_\mu f)(x) := \int f\,\mathrm{d}(\mu \star \varepsilon_x) = \langle f, \mu \star \varepsilon_x\rangle.$$

In particular, for $\mu = \varepsilon_{x_0}$ we use the abbreviations $R_{x_0} := R_{\varepsilon_{x_0}}$ resp. $L_{x_0} := L_{\varepsilon_{x_0}}$ for the right and left translations.

We collect some well-known *properties of convolution operators* which are easily verified and are tacitly used in the sequel. See e.g., [17, 22] (for groups), and [6], 1.2.15–1.2.18 (for hypergroups).

Proposition 2.1

(a) R_μ *and* L_μ *are bounded linear operators acting on* $C_0(\mathbb{G})$ *with* $\|R_\mu\|_\infty = \|L_\mu\|_\infty = \|\mu\|$.
(b) $R_\mu L_\nu = L_\nu R_\mu \ \forall \mu, \nu \in \mathcal{M}^b(\mathbb{G})$.
(c) $R_{\mu\star\nu} = R_\mu R_\nu$ *and* $L_{\mu\star\nu} = L_\nu L_\mu \ \forall \mu, \nu \in \mathcal{M}^b(\mathbb{G})$.
(d) $\langle f, \mu \star \nu\rangle = \langle R_\mu f, \nu\rangle = \langle L_\nu f, \mu\rangle \ \forall \mu, \nu \in \mathcal{M}^b(\mathbb{G}),\ f \in C_0(\mathbb{G})$.

In particular, for $\nu = \varepsilon_e$ *resp.* $= \varepsilon_{x_0}$ *we have*

(d1) $\langle f, \mu\rangle = R_\mu f(e) = L_\mu f(e) \ \forall \mu \in \mathcal{M}^b(\mathbb{G}),\ f \in C_0(\mathbb{G})$,
(d2) $f(x_0) = \langle f, \varepsilon_{x_0}\rangle = R_{x_0} f(e) = L_{x_0} f(e) \ \forall f \in C_0(\mathbb{G}),\ x_0 \in \mathbb{G}$,
(d3) $R_\mu f(x_0) = \langle f, \varepsilon_{x_0} \star \mu\rangle = \langle R_\mu f, \varepsilon_{x_0}\rangle = \langle L_{x_0} f, \mu\rangle = \langle L_{x_0} R_\mu f, \varepsilon_e\rangle \ \forall \mu \in \mathcal{M}^b(\mathbb{G}),\ f \in C_0(\mathbb{G}),\ x_0 \in \mathbb{G}$,
(d4) $L_\mu f(x_0) = \langle f, \mu \star \varepsilon_{x_0}\rangle = \langle L_\mu f, \varepsilon_{x_0}\rangle = \langle R_{x_0} f, \mu\rangle = \langle R_{x_0} L_\mu f, \varepsilon_e\rangle \ \forall \mu \in \mathcal{M}^b(\mathbb{G}),\ f \in C_0(\mathbb{G}),\ x_0 \in \mathbb{G}$.

Proposition 2.2 *Let* $f \in C_0(\mathbb{G})$, *and let* $x_0 \in \mathbb{G}$ *such that* $|f(x_0)| = \|f\|_\infty$. *Then* $\|f\|_\infty = |R_{x_0} f(e)| = \|R_{x_0} f\|_\infty$.

⟦R_{x_0} is a contraction (Proposition 2.1 (a)), hence $\|R_{x_0} f\|_\infty \le \|f\|_\infty$. On the other hand, according to property d2) in Proposition 2.1, $|f(x_0)| = |\langle R_{x_0} f, \varepsilon_e\rangle|$, whence $\|f\|_\infty = |f(x_0)| = |R_{x_0} f(e)| \le \|R_{x_0} f\|_\infty$.⟧

Let $T := R_\lambda,\ \lambda \in \mathcal{M}^b(\mathbb{G})$. T is left-invariant, i.e. $T L_x = L_x T \ \forall x \in \mathbb{G}$ (see Proposition 2.1) and $\langle f, \lambda\rangle = Tf(e),\ Tf(x) = \langle L_x f, \lambda\rangle$. This is a motivation to define

Definition 2.3 A subspace $\mathbb{D} \subseteq C_0(\mathbb{G})$ is called left- resp. right-invariant if $L_x\mathbb{D} \subseteq \mathbb{D}$ resp. $R_x\mathbb{D} \subseteq \mathbb{D}$, $\forall x \in \mathbb{G}$, and a linear operator $U : \mathbb{D} \to C_0(\mathbb{G})$ is called left invariant if $\mathbb{D}$ is left invariant and $UL_x = L_xU$ $\forall x \in \mathbb{G}$. (Hence $UL_\nu = L_\nu U$ for all $\nu \in \mathcal{M}^b(\mathbb{G})$ with $L_\nu(\mathbb{D}) \subseteq \mathbb{D}$.)

In this case, we define the linear functional $A : \mathbb{D} \to \mathbb{C}$ by $\langle f, A\rangle := Uf(e)$, hence (according to Proposition 2.1 (d2)) $Uf(x) = L_xUf(e) = UL_xf(e) = \langle L_xf, A\rangle$. This motivates the notation $U = R_A$ (in analogy to Proposition 2.1 (d3)).

Definition 2.4 Let $U : \mathbb{D} \to C_0(\mathbb{G})$ be a linear operator acting on a subspace $\mathbb{D} \subseteq C_0(\mathbb{G})$. U is called dissipative if for all $f \in \mathbb{D}$, for all $x_0 \in \mathbb{G}$ such that $f(x_0) = \|f\|_\infty$, it follows $\Re(Uf(x_0)) \leq 0$. ($\Re$ denoting the real part.) $(U, \mathbb{D})$ is m-dissipative if in addition U is closed and $(I - U)(\mathbb{D}) = C_0(\mathbb{G})$. See e.g., [16, 39].)

Proposition 2.5

(a) *Let $(T_t)_{t\geq 0}$ be a C_0-contraction semigroup on $C_0(\mathbb{G})$ with infinitesimal generator $(U := \frac{d^+}{dt}|_{t=0}T_t, D(U))$. Then the domain $D(U)$ is dense in $C_0(\mathbb{G})$ and U is m-dissipative.*

(b) *Conversely, let U be dissipative with dense domain $\mathbb{D}$. Then $(U, \mathbb{D})$ is closable, and the closure $(\overline{U}, \overline{\mathbb{D}})$ is closed and dissipative. Furthermore, $(I - U)(\mathbb{D})$ is dense in $(I - \overline{U})(\overline{\mathbb{D}})$.*

(c) *If in addition, $(I - U)(\mathbb{D})$ is dense in $C_0(\mathbb{G})$ then $(\overline{U}, \overline{\mathbb{D}})$ is the generator of a (uniquely determined) C_0-contraction semigroup $(T_t)_{t\geq 0}$.*

(*In the latter case, $\mathbb{D}$ is called ‘core’ for the generator of $(T_t)_{t\geq 0}$.*)

⟦*Theorem of Lumer-Phillips*, cf. [42], [16], I, Theorem 3.3.⟧

As a consequence of the Riesz representation theorem we obtain

Proposition 2.6 *A left invariant linear operator $T = R_A$—A as in Definition 2.3—defined on $\mathbb{D} := C_0(\mathbb{G})$ is the convolution operator of a bounded measure $A = \lambda \in \mathcal{M}^b(\mathbb{G})$, and conversely.*

Hence a C_0-semigroup of invariant operators on $C_0(\mathbb{G})$ is representable as $(T_t = R_{\lambda_t})_{t\geq 0}$, where $(\lambda_t)_{t\geq 0}$ is a continuous convolution semigroup in $\mathcal{M}^b(\mathbb{G})$ with $\lambda_0 = \varepsilon_e$ and $\|T_t\| = \|\lambda_t\|$. In particular, T_t are contractions iff $(\lambda_t) \subseteq \mathcal{M}^{(1)}(\mathbb{G})$.

⟦As well known, if (λ_t) is weakly continuous, then the operator semigroup $(R_{\lambda_t} = T_t)$ is continuous in the weak and hence also in the strong operator topology, and conversely.⟧

In the sequel we shall always tacitly assume for continuous convolution semigroups that $\lambda_0 = \varepsilon_e$.

Let $(\lambda_t)_{t\geq 0}$ be a continuous convolution semigroup in $\mathcal{M}^b(\mathbb{G})$ with corresponding C_0-operator semigroup $(T_t = R_{\lambda_t})_{t\geq 0}$. Then the infinitesimal generator $(U, D(U))$ is a left invariant operator. If moreover, $(\lambda_t) \subseteq \mathcal{M}^{(1)}(\mathbb{G})$ then $(U, D(U))$ is (left invariant and) dissipative. In view of Propositions 2.5 and 2.6 we have:

Proposition 2.7 *Let $(U, D(U))$ be left invariant and dissipative and assume $(I - U)D(U) = C_0(\mathbb{G})$, hence U is the generator of a C_0-contraction semigroup $(T_t)_{t\geq 0}$. Then $T_t = R_{\lambda_t}$ for some continuous convolution semigroup $(\lambda_t)_{t\geq 0} \subseteq \mathcal{M}^{(1)}(\mathbb{G})$.*

⟦For $\alpha > 0$ the resolvent $I_\alpha := (U - \frac{1}{\alpha}I)^{-1}$ is bounded, obviously left invariant, hence a convolution operator of a bounded measure. Any T_t is representable as limit of exponentials of resolvent operators, hence is itself left invariant. ([39], IX, §3 (1.17), [17], Hilfssatz 2.4.)⟧

Remark 2.8 Let $\mathbb{D}$ be a core for the generator of a semigroup of convolution operators $(R_{\lambda_t})_{t\geq 0}$. Then, by a slight abuse of language, we call $\mathbb{D}$ a core for the continuous convolution semigroup $(\lambda_t)_{t\geq 0}$.

Now we are ready to formulate the announced result of J. Faraut, K. Harzallah, F. Hirsch, J.P. Roth and M. Duflo [11, 12, 14, 15, 25–29, 46, 47]. We restrict to the case of continuous convolution semigroups with trivial idempotents $\lambda_0 = \varepsilon_e$. In the literature cited above, the results are mostly generalized to continuous convolution semigroups with non-trivial idempotents λ_0. (If $\lambda_t \geq 0$, then $\lambda_0 = \omega_K$, the Haar measure on some compact sub-(hyper)group K.)

Theorem 2.9 *Let $\mathbb{D}$ be a dense linear subspace of $C_0(\mathbb{G})$, $\mathbb{G}$ a locally compact group or a hypergroup.*

(a) *Assume* (i) $L_x\mathbb{D} \subseteq \mathbb{D}\ \forall x \in \mathbb{G}$ *and* (ii) $R_x\mathbb{D} \subseteq \mathbb{D}\ \forall x \in \mathbb{G}$.

Let $U : \mathbb{D} \to C_0(\mathbb{G})$ be a left invariant and dissipative linear operator. Then the closure $(\overline{U}, \overline{\mathbb{D}})$ is the generator of a left invariant contraction semigroup $(T_t = R_{\lambda_t})_{t\geq 0}$. I.e., $\mathbb{D}$ is a core for the continuous convolution semigroup $(\lambda_t) \subseteq \mathcal{M}^{(1)}(\mathbb{G})$.

(b) *More generally,* (ii) *may be replaced by* (ii′) $R_x\mathbb{D} \subseteq \overline{\mathbb{D}}\ \forall x \in \mathbb{G}$.

(c) *Let $(U, D(U))$ be a dissipative, closed and left invariant operator. Assume* (i) $\mathbb{D} \subseteq D(U)$ *to be left-invariant, and assume furthermore* (ii″) $R_x\mathbb{D} \subseteq D(U)$ $\forall x \in \mathbb{G}$.

Then $(U, D(U))$ is the generator of a left invariant contraction semigroup $(R_{\lambda_t})_{t\geq 0}$ and $\widetilde{\mathbb{D}} := \operatorname{span}\{R_x\mathbb{D} : x \in \mathbb{G}\}$ is a left- and right invariant core for $(U, D(U))$ (resp. for $(\lambda_t)_{t\geq 0}$).

(d) *Let $(U, D(U))$ be the generator of (R_{λ_t}). Let $\mathbb{D} \subseteq C_0(\mathbb{G})$ be a left-invariant $\|\cdot\|_\infty$-dense linear subspace. Let furthermore $\widetilde{\mathbb{D}}$ be a left and right invariant intermediate subspace with $\mathbb{D} \subseteq \widetilde{\mathbb{D}} \subseteq D(U)$. Let $|||\cdot|||$ be a norm on $\widetilde{\mathbb{D}}$ such that $\|\cdot\|_\infty \leq |||\cdot|||$. Assume finally that $\mathbb{D}$ is $|||\cdot|||$-dense in $\widetilde{\mathbb{D}}$ and $U : \mathbb{D} \to C_0(\mathbb{G})$ is $|||\cdot||| - \|\cdot\|_\infty$-continuous.*

Then

(d1) *$\widetilde{\mathbb{D}}$ is a left- and right-invariant core and*
(d2) *$\mathbb{D}$ is a core for $(U, D(U))$ resp. for (λ_t).*

Proof The following sketch of the proof of (a), (b) follows—with different notations—the lines of the proofs in [25], Théorème 1, [26]. See also [17], 0 §4. For hypergroups a proof (of (a)) is contained in the thesis [44], 5.26.

(a), (b) Condition (ii$'$) is weaker than (ii), hence (b) $\Rightarrow$ (a).

We first note that

1. Condition (i) implies $L_\nu \mathbb{D} \subseteq \overline{\mathbb{D}}\ \forall \nu \in \mathcal{M}^b(\mathbb{G})$. In fact, approximating ν by measures ν_n with finite supports such that $L_{\nu_n} \to L_\nu$ in the strong operator topology and observing $L_{\nu_n}\mathbb{D} \subseteq \mathbb{D}$ for all n, yields $L_{\nu_n} f \to L_\nu f$ for $f \in \mathbb{D}$, and furthermore, $UL_{\nu_n} f = L_{\nu_n} Uf \to L_\nu Uf$. Hence $L_\nu f \in \overline{\mathbb{D}}$ and $\overline{U} L_\nu f = L_\nu Uf$.

Analogously, $\forall g \in \overline{\mathbb{D}}$ we obtain $L_\nu g \in \overline{\mathbb{D}}$ and $\overline{U} L_\nu g = L_\nu \overline{U} g$. (This applies in particular for $f \in \mathbb{D}$, $g := R_{x_0} f \in \overline{\mathbb{D}}$.)

2. Let $\nu \in ((I-U)\mathbb{D})^\perp$. Since $(I-U)\mathbb{D}$ is dense in $(I-\overline{U})\overline{\mathbb{D}}$, we have $\nu \perp (I-\overline{U})\overline{\mathbb{D}}$.

Let $f \in \mathbb{D}$, let $x_0 \in \mathbb{G}$ such that $\|L_\nu f\|_\infty = |L_\nu f(x_0)| = |R_{x_0} L_\nu f(e)|$, i.e., for some c with $|c| = 1$ we have $\|L_\nu f\|_\infty = c \cdot L_\nu f(x_0)$. W.l.o.g. we may assume $c = 1$, else replace f by $c \cdot f$.

As $g := R_{x_0} f \in \overline{\mathbb{D}}$ by assumption (ii$'$) we have $0 = \langle (I-\overline{U})R_{x_0} f, \nu\rangle = \langle L_\nu (I-\overline{U})g, \varepsilon_e\rangle = L_\nu g(e) - \overline{U} L_\nu g(e) = R_{x_0} L_\nu f(e) - \overline{U} R_{x_0} L_\nu f(e) = \|R_{x_0} L_\nu f\|_\infty - \overline{U} R_{x_0} L_\nu f(e)$. Since $\|L_\nu f\|_\infty = (R_{x_0} L_\nu f)(e) = \|R_{x_0} L_\nu f\|_\infty$ (cf. Propositions 2.1 and 2.2) and $\overline{U}$ is dissipative, we have $\Re \overline{U} R_{x_0} L_\nu f(e) \le 0$. Therefore, $\|R_{x_0} L_\nu f\|_\infty = 0$. According to property (a) in Proposition 2.1, $\|L_\nu f\|_\infty = 0$ follows. Since $\mathbb{D}$ is dense in $C_0(\mathbb{G})$ we have proved $\nu = 0$.

3. Therefore, $(I-U)\mathbb{D}$ is dense in $C_0(\mathbb{G})$.

Assertion (b) (and hence (a)) follows by Proposition 2.7.

(c) Put $\widetilde{\mathbb{D}} := \operatorname{span}\{R_x \mathbb{D} : x \in \mathbb{G}\}$.

Claim: $\widetilde{\mathbb{D}}$ is a core for $(U, D(U))$. Hence by assumption, $(U, D(U))$ is m-dissipative and therefore a generator.

⟦Obviously, $\mathbb{D} \subseteq \widetilde{\mathbb{D}} \subseteq D(U)$. Hence $\widetilde{\mathbb{D}}$ is dense, by construction left and right invariant and therefore according to (a), $\widetilde{\mathbb{D}}$ is a core for the closure of the restriction $(U|_{\widetilde{\mathbb{D}}}, \widetilde{\mathbb{D}})$, the generator of an invariant semigroup. In particular, $(I-U)\widetilde{\mathbb{D}}$ is dense in $C_0(\mathbb{G})$. Since $(U, D(U))$ is closed, we observe $\overline{\widetilde{\mathbb{D}}} \subseteq D(U)$, hence $(\overline{U}, \overline{\widetilde{\mathbb{D}}}) = (U, D(U))$ and $(I-U)\overline{\widetilde{\mathbb{D}}} \subseteq (I-U)D(U) = C_0(\mathbb{G})$. Whence the assertion.⟧

(d) 1. The restriction $U|_{\widetilde{\mathbb{D}}}$ is uniquely determined by $U|_{\mathbb{D}}$. ⟦Let $f_n \in \mathbb{D}$, $f_n \to f \in \widetilde{\mathbb{D}}$ w.r.t. $|||\cdot|||$. (Hence also $\|f_n - f\|_\infty \to 0$.) Then $\|Uf_n - Uf\|_\infty \to 0$ by assumption, and hence $f \in \overline{\mathbb{D}}$. Therefore, $\overline{\mathbb{D}} \supseteq \widetilde{\mathbb{D}}$ and $U|_{\widetilde{\mathbb{D}}}$ coincides with $\overline{U}|_{\widetilde{\mathbb{D}}}$ on $\widetilde{\mathbb{D}}$. ($(\overline{U}, \overline{\mathbb{D}})$ denoting the closure of $(U|_{\mathbb{D}}, \mathbb{D})$.)⟧

2. $\widetilde{\mathbb{D}}$ is a core for $(U, D(U)$ by (a), (b)). Whence (d1) follows.

3. Let $g \in (I-U)(\widetilde{\mathbb{D}})$, $g = (I-U)f$ for some $f \in \widetilde{\mathbb{D}}$. By assumption, there exist $f_n \in \mathbb{D}$ with $|||f_n - f||| \to 0$, hence $\|(I-U)f_n - g\|_\infty \to 0$. Therefore, $(I-U)\mathbb{D}$ is $\|\cdot\|_\infty$-dense in $(I-U)\widetilde{\mathbb{D}}$, hence, by (d1), dense in $C_0(\mathbb{G})$. Whence (d2) is proved. □

Remark 2.10 Note that in general on (non-compact, non-Abelian) locally compact groups there exist left-invariant closed dissipative operators with dense left-invariant domain which are not generators of C_0-semigroups. See e.g. [45], Proposition 2.2. So additional conditions as in Theorem 2.9 (a)–(d) are needed.

We obtain immediately the well known result:

Corollary 2.11 *Let* $\mathbb{G}$ *be a locally compact group. Then the Schwartz-Bruhat test function space* $\mathcal{D}(\mathbb{G})$ *is a common core for all continuous convolution semigroups* $(\lambda_t)_{t\geq 0}$ *in* $\mathcal{M}^{(1)}(\mathbb{G})$, *in particular, for continuous convolution semigroups of probabilities.*

⟦$\mathcal{D}(\mathbb{G})$ is dense, left- and right-invariant and—according to the Lévy-Khinchin-Hunt representation—$\mathcal{D}(\mathbb{G})$ is contained in the domain of the generator of any continuous convolution semigroup in $\mathcal{M}^{(1)}(\mathbb{G})$. Cf. e.g., [22], 4.4.18, 4.5.8 for continuous convolution semigroups of probabilities, see e.g., [11–15, 17, 56, 57] for the more general case $\mathcal{M}^{(1)}(\mathbb{G})$. The assertion follows by Theorem 2.9 (a).⟧

Corollary 2.12 *Let* $\mathbb{G}$ *be an Abelian locally compact group or an Abelian hypergroup with dual* $\widehat{\mathbb{G}}$. *Then the space of 'analytic vectors'* $\mathcal{A} := (L^1_c(\widehat{\mathbb{G}}))^\vee$ *is a common core for all continuous convolution semigroups* $(\lambda_t)_{t\geq 0}$ *in* $\mathcal{M}^{(1)}(\mathbb{G})$. (*Here* L^1_c *denotes the space of functions with compact support which are integrable on the dual* $\widehat{\mathbb{G}}$ *w.r.t. the Haar resp. Plancherel measure, and* $^\vee$ *denotes the inverse Fourier transform.*) *Analogously,* $C_c(\widehat{\mathbb{G}})^\vee$ *and* $L^{(2)}_c(\widehat{\mathbb{G}})^\vee$ *share this property.*

⟦$\mathcal{A}$ is dense and left- (hence trivially right)-invariant. Furthermore, for any $f \in \mathcal{A}$ and any continuous convolution semigroup (λ_t) we observe that $t \mapsto R_{\lambda_t} f = (\widehat{\lambda_t} \cdot \widehat{f})^\vee = (\mathrm{e}^{t\cdot\psi} \cdot \widehat{f})^\vee$ is analytic (where the logarithm ψ is defined by $\widehat{\lambda_t} = \mathrm{e}^{t\cdot\psi}$). Therefore in particular, f is contained in the domain of the generator.

For groups a proof is found in e.g. [9], for hypergroups see [44], 5.17, 5.22. The assertion follows by Theorem 2.9 (a).

Alternatively: it is well known that a dense subspace of analytic vectors is a core. (Cf. [16], Chap. 1, Theorem 9.20.)⟧

Remark 2.13 For later use we note that the cores $\mathcal{D}(\mathbb{G})$ and $\mathcal{A}$ constructed above in Corollary 2.11 resp. 2.12 are (left- and right-invariant and) invariant under topological automorphisms of $\mathbb{G}$.

3 Semi-direct products $\Gamma = \mathbb{G} \rtimes \mathbb{R}$: The case of locally compact groups $\mathbb{G}$

Throughout in this section let $\mathbb{G}, \mathbb{G}_i$ denote locally compact topological groups. First we note a further corollary to Theorem 2.9:

Corollary 3.1 (Direct products) *Let* $\mathbb{G}_i$, $i = 1, 2$, *be locally compact groups with test function spaces* $\mathcal{D}(\mathbb{G}_1), \mathcal{D}(\mathbb{G}_2)$ *respectively. Then the subspace* $\mathbb{D} := \mathcal{D}(\mathbb{G}_1) \otimes \mathcal{D}(\mathbb{G}_2) \subseteq \mathcal{D}(\mathbb{G}_1 \otimes \mathbb{G}_2)$ *is a common core for continuous convolution semigroups in* $\mathcal{M}^{(1)}(\mathbb{G}_1 \otimes \mathbb{G}_2)$.

⟦On the one hand, $\mathbb{D} \subseteq \mathcal{D}(\mathbb{G}_1 \otimes \mathbb{G}_2) \subseteq D(U)$ for any generator $(U, D(U))$ of a continuous convolution semigroup, as mentioned in Corollary 2.11. On the other hand, $\mathbb{D}$ satisfies the conditions (i) and (ii) of Theorem 2.9 (a).⟧

$\mathbb{D}$ is dense in $C_c(\mathbb{G}_1 \otimes \mathbb{G}_2)$. Furthermore, we note a well-known result:

Lemma 3.2 *Let* $\mathbb{G}_i, i = 1, 2$ *be Lie groups. Then* $\mathcal{D}(\mathbb{G}_1) \otimes \mathcal{D}(G_2)$ *is dense in* $\mathcal{D}(\mathbb{G}_1 \otimes \mathbb{G}_2)$ *w.r.t.* $|||\cdot||| := \|\cdot\|_{C^{(2)}(\mathbb{G}_1 \otimes \mathbb{G}_2)}$.

Proof We sketch a proof:

1. Obviously it suffices to show for sufficiently small neighborhoods V_i that $C_c^\infty(V_1) \otimes C_c^\infty(V_2)$ is dense in $C_c^\infty(V_1 \otimes V_2)$ w.r.t. $\|\cdot\|_{C^{(2)}}$. Hence, as $\mathbb{G}_i$ are Lie groups, we may assume $\mathbb{V}_1$ and V_2 to be neighborhoods of 0 in the tangent spaces $\mathbb{R}^{d_1}$ and $\mathbb{R}^{d_2}$ respectively.

2. Therefore it is sufficient to show that $C_c^\infty(\mathbb{R}^{d_1}) \otimes C_c^\infty(\mathbb{R}^{d_2})$ is dense in $C_c^\infty(\mathbb{R}^{d_1+d_2})$.

3. For Hermite functions $h^{(i)} \in \mathcal{S}(\mathbb{R}^1)$ we have $h^{(1)} \otimes h^{(2)}(x, y) = p(x, y) \cdot \mathrm{e}^{-x^2}\mathrm{e}^{-y^2}$ for some polynomial p. Hence, if $\mathcal{H}(\mathbb{R}^d)$ denotes the vector space generated by Hermite functions on $\mathbb{R}^d$, then $\mathcal{H}(\mathbb{R}^{d_1}) \otimes \mathcal{H}(\mathbb{R}^{d_2}) = \mathcal{H}(\mathbb{R}^{d_1+d_2})$.

4. Using the facts that $\mathcal{H}(\mathbb{R}^d)$ is dense in the Schwartz function space $\mathcal{S}(\mathbb{R}^d)$, and that any function $f \in \mathcal{D} = \mathbb{C}_c^\infty(\mathbb{R}^d)$ can be approximated by $f_n \in \mathcal{S}(\mathbb{R}^d)$ (w.r.t. the natural topologies on $\mathcal{S}$ and $\mathcal{D}$), the assertion is easily proved. ☐

Now let $\mathbb{G}$ denote a locally compact group and let $(T_t)_{t\in\mathbb{R}} \subseteq \mathrm{Aut}(\mathbb{G})$ be a continuous one parameter group. The semi-direct product $\Gamma = \mathbb{G} \rtimes \mathbb{R}$ is the Cartesian product $\mathbb{G} \otimes \mathbb{R}$ equipped with the group operation $(x, s)(y, t) := (xT_s(y), s + t)$. Γ is a locally compact group and hence $\mathcal{D}(\Gamma)$ is a common core for continuous convolution semigroups in $\mathcal{M}^{(1)}(\Gamma)$.

Proposition 3.3 (Semi-direct products $\Gamma = \mathbb{G} \rtimes \mathbb{R}$) *Let* $\mathbb{G}$ *be a Lie group. Then* $\mathbb{D} := \mathcal{D}(\mathbb{G}) \otimes \mathcal{D}(\mathbb{R}) \subseteq \mathcal{D}(\Gamma)$ *is a common core for continuous convolution semigroups in* $\mathcal{M}^1(\Gamma)$.

Proof In contrast to the above mentioned Corollary 3.1 now the proof relies on the weaker assumptions in Theorem 2.9 (d).

$\mathbb{D}$ is obviously left invariant: For $\varphi \otimes \psi \in \mathcal{D}(\mathbb{G}) \otimes \mathcal{D}(\mathbb{R})$ we have

$$L_{(y,t)}(\varphi \otimes \psi)(x, s) = \varphi(yT_t(x)) \cdot \psi(s + t) =: \varphi_1(x) \cdot \psi_1(s)$$

with $\varphi_1 \in \mathcal{D}(\mathbb{G}_1)$, $\psi_1 \in \mathcal{D}(\mathbb{R})$. Hence $L_{(y,t)}(\varphi \otimes \psi) \in \mathbb{D}\ \forall (y, t) \in \Gamma$.

Furthermore, according to Lemma 3.2, $\mathbb{D}$ is $|||\cdot|||$-dense in $\mathcal{D}(\Gamma)$. Put $\widetilde{\mathbb{D}} := \mathrm{span}\{R_{(y,t)}\varphi \otimes \psi : (y, t) \in \Gamma, \varphi \in \mathcal{D}(\mathbb{G}), \psi \in \mathcal{D}(\mathbb{R})\}$, a subspace of $\mathcal{D}(\Gamma)$. Note that $R_{(y,t)}\varphi \otimes \psi(x, s) = \varphi(xT_s(y)) \cdot \psi(s + t)$. Let (λ_t) be a continuous convolution semigroup with generator $(U = R_A, D(U))$. Hence $\mathbb{D} \subseteq \widetilde{\mathbb{D}} \subseteq \mathcal{D}(\Gamma) \subseteq D(U)$.

$\widetilde{\mathbb{D}}$ is obviously left- and right-invariant, hence a core for $(U, D(U))$ according to Theorem 2.9 (a), resp. Corollary 2.11.

The Lévy-Khinchin-Hunt representation (cf. e.g., [22], 4.4.18, 4.5.8, [23] for $\mathcal{M}^1(\mathbb{G})$, resp. [11, 13–15, 17, 56, 57] for $\mathcal{M}^{(1)}(\mathbb{G})$) yields that the restriction of the generator $U = R_A : (C_c^{(2)}(\Gamma), \|\cdot\|_{C^{(2)}}) \to (C_0(\Gamma), \|\cdot\|_\infty)$ is continuous.

Whence the assertion follows by Theorem 2.9 (d). ☐

In all examples we have in mind, the underlying group is a (simply connected, nilpotent) Lie group. Nevertheless it is worth to point out that this result is true for general locally compact groups $\mathbb{G}$ which admit a continuous one-parameter group of automorphisms $(T_t)_{t\in\mathbb{R}} \subseteq \mathrm{Aut}(\mathbb{G})$:

Theorem 3.4 *Let $\mathbb{G}$ be a locally compact group with $(T_t)_{t\in\mathbb{R}} \subseteq \mathrm{Aut}(\mathbb{G})$, assume that (T_t) is not relatively compact, hence $\cong \mathbb{R}$. We define as above the semi-direct extension $\Gamma = \mathbb{G} \rtimes \mathbb{R}$ and put again $\mathbb{D} := \mathcal{D}(\mathbb{G}) \otimes \mathcal{D}(\mathbb{R})$.*

Let $(\lambda_t)_{t\geq 0} \subseteq \mathcal{M}^{(1)}(\Gamma)$ be a continuous convolution semigroup with generating functional A resp. infinitesimal generator $(U = R_A, D(U))$. Then $\mathbb{D}$ is a core for $(\lambda_t)_{t\geq 0}$.

Proof We sketch a proof:

$\mathbb{D}$ is dense in $C_0(\Gamma)$ and $\mathbb{D} \subseteq \mathcal{D}(\Gamma) \subseteq D(U)$. As before, it follows immediately that $\mathbb{D}$ is left invariant. Let, as in Proposition 3.3, $\widetilde{\mathbb{D}}$ denote the linear span of $\{R_{(y,t)}\varphi \otimes \psi : \varphi \in \mathcal{D}(\mathbb{G}), \psi \in \mathcal{D}(\mathbb{R})\}$. (T_t) is a continuous one-parameter group. The connected component $\mathbb{G}_0$ is characteristic and $\mathbb{G}/\mathbb{G}_0$ is totally disconnected. Therefore, the induced automorphisms $\overline{T}_t$ act trivially on $\mathbb{G}/\mathbb{G}_0$.

Choose an open almost connected subgroup $\mathbb{G}_1 \subseteq \mathbb{G}$, i.e., such that $\mathbb{G}_1/\mathbb{G}_0$ is compact. Then, $\Gamma_1 := \mathbb{G}_1 \rtimes \mathbb{R}$ is an open almost connected subgroup of Γ, therefore (e.g., see [21], 3.1.22), we have $\Gamma_1 = \lim_{\leftarrow} \Gamma_1/L^\alpha$ with compact normal subgroups $L^\alpha \subseteq \Gamma_1$. Hence there exist (T_t)-invariant normal subgroups $K^\alpha \subseteq \mathbb{G}_1$ with $L^\alpha = K^\alpha \otimes \{0\}$, and $\mathbb{G}_1 = \lim_{\leftarrow} \mathbb{G}_1/K^\alpha$.

The Lévy-Khinchin-Hunt representation for general locally compact groups (cf. e.g., [22, 23] for probabilities, resp. [11, 13–15, 17, 56, 57] for $\mathcal{M}^{(1)}(\mathbb{G})$) yields that $A = B + \gamma$, where γ is a bounded measure (a Poisson generator), and B is supported by Γ_1. We have $U = R_A = R_B + R_\gamma$ and, as γ is bounded, it is easily seen (developing $(I - R_B - R_\gamma)^{-1}$ in a Neumann series) that $(I - R_A)\mathbb{D}$ is dense iff $(I - R_B)\mathbb{D}$ is. Hence w.l.o.g. we may assume $\mathbb{G} = \mathbb{G}_1$ and $\Gamma = \Gamma_1$ to be Lie-projective.

Let, as in Proposition 3.3, $\varphi \otimes \psi \in \mathbb{D} = \mathcal{D}(\mathbb{G}) \otimes \mathcal{D}(\mathbb{R})$ and $R_{(y,t)}\varphi \otimes \psi =: h \in \widetilde{\mathbb{D}}$. Since $\mathbb{D} \subseteq \widetilde{\mathbb{D}} \subseteq \mathcal{D}(\Gamma)$ there exist α such that $\varphi \otimes \psi$ and h are L^α-invariant.

Hence w.l.o.g. we may assume Γ and $\mathbb{G}$ to be Lie groups. Thus the proof is reduced to the case handled in Proposition 3.3. □

3.1 Lie-Trotter formulas

We recall Lie-Trotter product formulas for addition of generators of C_0-semigroups and its applications to continuous convolution semigroups. For the background see e.g., P.R. Chernoff [7, 8], Theorem 1.1, [16], I, Theorem 8.12, Theorem 8.4, and the literature mentioned there. For applications to continuous convolution semigroups see e.g., [17], I, §2.4.

Proposition 3.5

(a1) (Lie-Trotter formula) *The sum $U + V$ of generators of C_0-contraction semigroups $(U, D(U))$ and $(V, D(V))$ defines a dissipative operator on $D(U) \cap D(V)$. If $D(U) \cap D(V)$ is a core for the closure $\overline{U + V}$ and if $\overline{U + V}$ is the* *generator of a contraction semigroup, then the involved semigroups are related by the Lie-Trotter formula*:

$$\mathrm{e}^{t(\overline{U+V})} = \lim_{n\to\infty} (\mathrm{e}^{(t/n)U} \mathrm{e}^{(t/n)V})^n \tag{3.1}$$

The limit exists in the strong operator topology, uniformly on compact subsets of $\mathbb{R}_+$. (*For short we write* $\mathrm{e}^{t(U+V)}$ *instead of* $\mathrm{e}^{t(\overline{U+V})}$ *then.*)

(a2) (Chernoff's product formula) *Let* $\mathbb{R}_+ \ni t \mapsto F(t)$ *be a contraction valued map, let* $\mathbb{D}$ *be a dense subspace, such that for all* $x \in \mathbb{D}$ $Ux := \frac{\mathrm{d}^+}{\mathrm{d}t}|_{t=0}\, F(t)(x)$ *exists, and assume that the closure* $(\overline{U}, \overline{\mathbb{D}})$ *generates a* C_0*-contraction semigroup* (e^{tU}). *Then* $\mathrm{e}^{tU} = \lim_{n\to\infty} F(t/n)^n$ *in the strong operator topology, uniformly on compact subsets of* $\mathbb{R}_+$.

(b) *Applying* (*a*1) *and* (*a*2) *to continuous convolution semigroups* (*resp. to the corresponding convolution operators*) *on locally compact groups or on hypergroups we obtain* (*in view of the existence of a common core*):

(b1) *Let* $(\mu_t)_{t\geq 0}$, $(\nu_t)_{t\geq 0} \subseteq \mathcal{M}^{(1)}(\mathbb{G})$ *be continuous convolution semigroups on a locally compact group or hypergroup* $\mathbb{G}$. *Let* $\mathbb{D}$ *be a common core for both continuous convolution semigroups* (*e.g.,* $\mathbb{D} = \mathcal{D}(\mathbb{G})$ *in case of locally compact groups*). *Then the sum of the generators is at least defined on* $\mathbb{D}$ *and its closure generates a continuous convolution semigroup* $(\lambda_t)_{t\geq 0}$. *Furthermore, the Lie-Trotter formula for continuous convolution semigroups holds true*:

$$\lambda_t = \lim_{n\to\infty} (\mu_{t/n} \star \nu_{t/n})^n. \tag{3.2}$$

(b2) *Let the mapping* $\mathbb{R}_+ \ni t \mapsto \lambda(t) \in \mathcal{M}^{(1)}(\mathbb{G})$, $\lambda(0) = \varepsilon_e$, *be differentiable for all* $f \in \mathbb{D}$, $\langle A, f\rangle := \frac{\mathrm{d}^+}{\mathrm{d}t}|_{t=0}\, \langle \lambda(t), f\rangle$, *where* $\mathbb{D}$ *is a left-invariant core for the continuous convolution semigroup* (λ_t) *generated by* A *resp. by* R_A. *Then, by Chernoff's product formula* (*a*2) *we have*:

$$\lambda_t = \lim_{n\to\infty} (\lambda(t/n))^n \tag{3.3}$$

uniformly on compact subsets of $\mathbb{R}_+$.

4 The main results

As before let $\mathbb{G}$ be a locally compact group or a hypergroup and $\mathbb{T} = (T_t) \subseteq \mathrm{Aut}(\mathbb{G})$ a continuous one-parameter group of automorphisms defining the semi-direct extension $\Gamma := \mathbb{G} \rtimes \mathbb{R}$. In the following we consider a sub-semigroup of $\mathcal{M}^1(\Gamma)$, defined as $\mathcal{M}^1_*(\Gamma) := \{\mu \otimes \varepsilon_t : \mu \in \mathcal{M}^1(\mathbb{G}), t \in \mathbb{R}\}$. (Analogously $\mathcal{M}^{(1)}_*(\Gamma)$, $\mathcal{M}^{(1)}_{*,+}(\Gamma)$, $\mathcal{M}^b_*(\Gamma)$ etc. are defined.)

Recall the definition of a *M-semigroup* (also called *skew convolution semigroup* or solution of a *cocycle equation*) in the Introduction: A continuous one-parameter family $(\mu(t))_{t\geq 0} \subseteq \mathcal{M}^1(\mathbb{G})$ is a M-semigroup or skew convolution semigroup if

$$\mu(s+t) = \mu(s) \star T_s(\mu(t)) \quad \text{for all } s, t \geq 0. \tag{4.1}$$

As immediately seen, the family $(\mu(t))_{t\geq 0}$ is a M-semigroup in $\mathcal{M}^1(\mathbb{G})$ iff $(\lambda_t := \mu(t)\otimes\varepsilon_t)_{t\geq 0}$ is a continuous convolution semigroup in $\mathcal{M}^1_*(\Gamma)$. Furthermore, as immediately verified, for $f\in\mathbb{D} := \mathcal{D}(\mathbb{G})\otimes\mathcal{D}(\mathbb{R})$, the generator U of (R_{λ_t}) splits as $Uf = (W+P)f$ resp. $Wf = (U-P)f$, with $Wf = \frac{\mathrm{d}^+}{\mathrm{d}t}|_{t=0}R_{\mu(t)\otimes\varepsilon_0}f$ and $\pm Pf = \frac{\mathrm{d}^+}{\mathrm{d}t}|_{t=0}R_{\varepsilon_e\otimes\varepsilon_{\pm t}}$. W and $\pm P$—by construction dissipative left invariant operators—are extended to generators of continuous convolution semigroups $(\sigma_t := \mu_t\otimes\varepsilon_0)_{t\geq 0}$ and $(p_t^{\pm} := \varepsilon_e\otimes\varepsilon_{\pm t})_{t\geq 0}$ respectively (cf. Theorem 3.4). The Lie-Trotter formula (3.2) yields $\lambda_t = \lim_n(\sigma_{t/n}\star p^+_{t/n})^n$, $\sigma_t = \lim_n(\lambda_{t/n}\star p^-_{t/n})^n$. Considering the projection $\Gamma\to\mathbb{G}$ yields the forward and backward Lie Trotter formulas $(LT1)$ resp. $(LT2)$ mentioned in the Introduction, cf. (1.1). Hence we obtain a bijection $(\mu(t))_{t\geq 0}\leftrightarrow(\mu_t)_{t\geq 0}$.

Summarizing, the steps in Sect. 3 yield the following result, cf. e.g., [21], 2.14 III, [19], Theorem C; see also e.g., [3, 18] for applications:

Theorem 4.1 *Let $\mathbb{G}$ be a locally compact group and $\mathbb{T} := (T_t)_{t\geq 0}\subseteq \mathrm{Aut}(\mathbb{G})$ a fixed continuous one-parameter group. Furthermore, let $\Gamma := \mathbb{G}\rtimes\mathbb{R}$ denote the semi-direct extension of $\mathbb{G}$ defined by $\mathbb{T}$. Then*

(a) *$\mathbb{D} := \mathcal{D}(\mathbb{G})\otimes\mathcal{D}(\mathbb{R})$ is a core for any continuous convolution semigroup of probabilities in $M_*^{(1)}(\Gamma)$.*
(b) *There exists a bijection $(\mu(t))_{t\geq 0}\leftrightarrow(\mu_t)_{t\geq 0}$ between M-semigroups and continuous convolution semigroups in $\mathcal{M}^1(\mathbb{G})$ resp. in $\mathcal{M}^{(1)}(\mathbb{G})$, i.e., between* (*distributions of*) *Ornstein-Uhlenbeck processes and* (*background driving*) *Lévy processes. The bijection is expressed by the forward and backward Lie-Trotter formulas* $(LT1)$ *resp.* $(LT2)$ (*cf.* (1.1)).

For (matrix cone-) hypergroups we shall prove in analogy to the group case:

Theorem 4.2 *Let $\mathcal{K}$ be a matrix cone hypergroup* (*investigated in* [48, 53]) *with fixed continuous one parameter group $\mathbb{T} := (T_t)_{t\geq 0}\subseteq\mathrm{Aut}(\mathcal{K})$. Define the semi-direct hypergroup-product $\Gamma := \mathcal{K}\rtimes\mathbb{R}$ in canonical way.*

Then the assertions (a) *and* (b) *of Theorem 4.1 hold true in this situation, where $\mathcal{D}(\mathbb{G})$ and $\mathbb{D}$ have to be replaced by suitable function spaces $\mathcal{A}$ and $\widetilde{\mathbb{D}}$* (*defined in the proof of Theorem 5.21 and in Definition 5.23 below*) *on the hypergroups $\mathcal{K}$ and Γ respectively.*

In particular, $\widetilde{\mathbb{D}}$ is again a common core for all continuous convolution semigroups in $\mathcal{M}^1_*(\Gamma)$.

The proof of Theorem 4.1, worked out in Sect. 3, relied mainly on Theorem 2.9. In fact, Theorem 4.1, in particular (a), is well-known and was used several times—at least in the case of Lie groups—without pointing out that the original version of Theorem 2.9 (a) needs a straight forward generalization (i.e. Theorem 2.9 (b)–(d)) to handle the case of semi-direct products. (See e.g. [21], §2.14, [19].) We included a proof in order to show the differences to the case of hypergroups:

The proof of Theorem 4.2 is more complicated and not straight forward. In fact, the details are quite technical, but I was unable to find a better way. The proof will be carried out in Sect. 5, in a series of propositions, which may be interesting in their own right. At first, here we sketch an *outline of the proof*:

1. Assume $(\mu(t))_{t\geq 0}$ to be a M-semigroup on $\mathcal{K}$ with corresponding space-time semigroup (λ_t) in $\mathcal{M}^1_*(\Gamma)$. Then we construct a suitable core $\mathcal{E}$ for (λ_t) such that on $\mathcal{E}$ the generator U of the convolution operators (R_{λ_t}) splits $U = W + P$, W generating a continuous convolution semigroup $(\sigma_t = \mu_t \otimes \varepsilon_0)_{t\geq 0}$ concentrated on $\mathcal{K} \otimes \{0\} \cong \mathcal{K}$, and $\pm P$ generating the semigroups of shifts $(p_t^{\pm} := \varepsilon_{(e,\pm t)})_{t\geq 0}$ on $\{e\} \otimes \mathbb{R} \cong \mathbb{R}$. (Note that the constructed core $\mathcal{E}$ still depends on (λ_t).)

2. Then the Lie-Trotter formula (Proposition 3.5 (a1), (b1), (3.1)) applied to $U = W + P$ yields $(LT1)$ (as in the group case): Projecting to $\mathcal{K}$, the mapping $(\mu(t))_{t\geq 0} \mapsto (\mu_t)_{t\geq 0}$ is established.

3. Conversely, let (μ_t) be a continuous convolution semigroup on a matrix cone hypergroup $\mathcal{K}$. On Abelian hypergroups, hence on matrix cone hypergroups, there exists a subspace $\mathcal{A} \subseteq C_0(\mathcal{K})$ which is a common core for all continuous convolution semigroups on $\mathcal{K}$ and is invariant under shifts and automorphisms. (Cf. Corollary 2.12, resp. Remark 2.13.) By means of $\mathcal{A}$ we construct a subspace $\widetilde{\mathbb{D}} \subseteq C_0(\Gamma)$ which is a common core for continuous convolution semigroups in $\mathcal{M}^1_*(\Gamma)$.

4. Furthermore, let V be the generator of $(\mu_t)_{t\geq 0}$, let $(\sigma_t := \mu_t \otimes \varepsilon_0)_{t\geq 0}$ with generator W, and let P as above, then $U = W + P$ is (the restriction to $\widetilde{\mathbb{D}}$ of) the generator of a continuous convolution semigroup $(\lambda_t = \mu(t) \otimes \varepsilon_t)_{t\geq 0} \subseteq \mathcal{M}^1_*(\Gamma)$. Applying the Lie-Trotter formulas to $U = W + P$ resp. $W = U - P$, and considering the space component, i.e., the projection to $\mathcal{K}$, we obtain $(LT1)$ and $(LT2)$ respectively.

5. Together with step 1 this yields the bijection $(\mu(t))_{t\geq 0} \leftrightarrow (\mu_t)_{t\geq 0}$ and the existence of a common core as asserted.

5 Semidirect products $\Gamma = \mathcal{K} \rtimes \mathbb{R}$: the case of matrix cone hypergroups $\mathcal{K}$

As announced in Theorem 4.2, our aim is to establish a 1-1-correspondence between M-semigroups on the one side and continuous convolution semigroups on the other for a class of hypergroups with ‘group-like behavior’: Such hypergroups on the cone of non-negative definite matrices $\mathcal{K}$ were recently investigated, cf. [48, 53], a class of hypergroups which share many features with locally compact groups. In particular, the group of automorphisms is well known, and there exist continuous one-parameter groups of automorphisms in abundance. (See e.g. [20] for an overview of some probabilistic structures on these hypergroups: In particular, the first section there contains a collection of basic properties which are needed here.) In the sequel we have these examples in mind, but results and proofs depend only on particular properties of $\mathcal{K}$, thus could be generalized to larger classes of hypergroups. Recently M. Voit [55] investigated ‘Heisenberg-like’ hypergroup structures on $\mathcal{K} \otimes \mathbb{R}$. For those non-commutative convolution structures, one parameter groups of automorphisms exist. It is reasonable to expect that the results can be partially generalized to these new classes of examples. (The investigations will be continued.)

Definition 5.1 Let $\mathcal{K}$ be the cone of positive semidefinite real $d \times d$-matrices endowed with a hypergroup structure (investigated in [48, 53]). (We restrict for convenience to the case of real matrices.) $\mathcal{K}$ is a commutative Hermitean hypergroup, furthermore, self-dual (i.e., $\widehat{\mathcal{K}}$ is a hypergroup $\cong \mathcal{K}$), with Pontryagin and Godement property. In particular, Lévy's continuity theorem is valid. $\mathcal{K}$ is aperiodic, i.e., without compact sub-hypergroups except the unit $\{e\}$. The unit of the hypergroup $\mathcal{K}$ is the zero-matrix, denoted by e.

Automorphisms of $\mathcal{K}$ are obtained in the following way: $\mathcal{K}$ is considered as subset of the $d \times (d+1)/2$-dimensional vector space $\mathbb{H} := \mathcal{K} - \mathcal{K}$ of (real) Hermitean matrices. For $a \in \mathrm{GL}(\mathbb{R}^d)$ put $T_a : \mathbb{H} \ni \kappa \mapsto ((a\kappa)(a\kappa)^*)^{1/2} = (a\kappa^2 a^*)^{1/2} \in \mathcal{K} \subseteq \mathbb{H}$. The restriction to $\mathcal{K}$ defines an hypergroup automorphism of $\mathcal{K}$. Let $(T_t)_{t\in\mathbb{R}}$ be a continuous one-parameter group in $\mathrm{Aut}(\mathcal{K})$. Then there exists a continuous one-parameter group $(a_t = \exp(tQ))_{t\in\mathbb{R}} \subseteq \mathrm{GL}(\mathbb{R}^d)$ such that $T_t = T_{a_t}$ $\forall t \in \mathbb{R}$. And conversely, $(T_{a_t}) \subseteq \mathrm{Aut}(\mathcal{K})$ for any one-parameter group (a_t). In the following we fix $(T_t := T_{a_t})$ with $(a_t = \exp t \cdot Q)_{t\in\mathbb{R}}$.

Let $\mathbb{V} := \mathbb{H} \otimes \mathbb{R}$, the Cartesian product, containing $\Gamma := \mathcal{K} \otimes \mathbb{R}$ as a subset. Γ, endowed with a convolution structure $\varepsilon_{(x,s)} * \varepsilon_{(y,t)} := (\varepsilon_x \star \varepsilon_{T_s(y)}) \otimes \varepsilon_{s+t}$ for $(x,s), (y,t) \in \Gamma$ and with involution defined by $(x,s)^- = (T_{-s}(x)^-, -s)$ is a (non commutative) hypergroup. (The axioms are easily verified. Note that in our case, $\mathcal{K}$ is Hermitean, hence in particular, $T_{-s}(x)^- = T_{-s}(x)$.) Therefore, the notation $\Gamma =: \mathcal{K} \rtimes \mathbb{R}$ is justified. As above, convolution in $\mathcal{K}$ and Γ will be denoted by '$\star$' and '$*$' respectively.

Probabilities on $\mathcal{K}$ resp. on Γ act by convolution on $C_0(\mathcal{K})$ and $C_0(\Gamma)$ respectively. (Cf. Proposition 2.1.) We denote the left and right convolution operators as follows: Let $f \in C_0(\mathcal{K})$, $g \in C_0(\Gamma)$, $z \in \mathcal{K}$ and $(z,r) \in \Gamma$.

$$\overset{\bullet}{R}_z f(x) := f(x \star z) := \int_{\mathcal{K}} f(y)\,\mathrm{d}(\varepsilon_x \star \varepsilon_z)(y), \tag{5.1}$$

$$\overset{\bullet}{L}_z f(x) := f(z \star x) := \int_{\mathcal{K}} f(y)\,\mathrm{d}(\varepsilon_z \star \varepsilon_x)(y). \tag{5.2}$$

$$R_{(z,r)} g(x,s) := g((x,s) * (z,r)) := \int_{\Gamma} g(y,u)\,\mathrm{d}(\varepsilon_{(x,s)} * \varepsilon_{(z,r)})(y,u),$$

$$L_{(z,r)} g(x,s) := g((z,r) * (x,s)) := \int_{\Gamma} g(y,u)\,\mathrm{d}(\varepsilon_{(z,r)} * \varepsilon_{(x,s)})(y,u).$$

In an analogous way we define for measures λ on Γ resp. μ on $\mathcal{K}$ the left resp. right convolution operators $\overset{\bullet}{R}_\mu$, $\overset{\bullet}{L}_\mu$, R_λ, L_λ on $C_0(\mathcal{K})$ and $C_0(\Gamma)$ respectively.

Definition 5.2 In the following we restrict again our considerations to measures on the 'space-time building' $\Gamma = \mathcal{K} \rtimes \mathbb{R}$ of 'space-time'-form $\lambda \in \mathcal{M}^1_*(\Gamma) := \{\mu \otimes \varepsilon_u : \mu \in \mathcal{M}^1(\mathcal{K}), u \in \mathbb{R}\}$. In that case we have

$$R_{\mu\otimes\varepsilon_u} g(x,s) = \int_{\mathcal{K}} g(x \star T_s(y), s+u)\,\mathrm{d}\mu(y), \tag{5.3}$$

$$L_{\mu\otimes\varepsilon_u} g(x,s) = \int_{\mathcal{K}} g(y \star T_u(x), s+u)\,\mathrm{d}\mu(y). \tag{5.4}$$

Note that for $g = \varphi \otimes \psi$ we obtain (with $\psi_u : s \mapsto \psi(s+u)$):

$$R_{\mu\otimes\varepsilon_u} g(x,s) = \int_{\mathcal{K}} \varphi(x \star T_s(y))\, \mathrm{d}\mu(y) \cdot \psi_u(s) = (\overset{\bullet}{R}_{T_s(\mu)}\, \varphi)(x) \cdot \psi_u(s),$$

$$L_{\mu\otimes\varepsilon_u} g(x,s) = \int_{\mathcal{K}} \varphi(y \star T_u(x))\, \mathrm{d}\mu(y) \cdot \psi_u(s) = (\overset{\bullet}{L}_{\mu}\, \varphi)(T_u(x)) \cdot \psi_u(s).$$

The involution on Γ induces involutions on spaces of functions and measures:
Let $g \in \mathbb{C}^b(\Gamma)$. Then $\widetilde{g}(x,s) := g((x,s)^-) = g(T_{-s}(x), -s)$.
Let $\lambda \in \mathcal{M}^b(\Gamma)$. Then $\int_\Gamma f\, \mathrm{d}\widetilde{\lambda} := \int_\Gamma \widetilde{f}\, \mathrm{d}\lambda$.
In particular, for $\lambda = \mu \otimes \varepsilon_u$ we obtain $\widetilde{\lambda} = T_{-u}(\mu) \otimes \varepsilon_{-u}$.

Proposition 5.3

(a) *For* $\lambda, \mu \in \mathcal{M}^b(\Gamma)$ *we have* $\widetilde{\lambda * \mu} = \widetilde{\mu} * \widetilde{\lambda}$.
(b) *For* $\lambda \in \mathcal{M}^b(\Gamma)$, $f \in C_0(\Gamma)$ *we have* $\widetilde{R_\lambda f} = L_{\widetilde{\lambda}} \widetilde{f}$.

We recall the notations of left invariant operators and subspaces introduced in Sect. 3; we have to distinguish between invariant operators on $\mathcal{K}$ and on the non-commutative hypergroup Γ.

The existence of background driving Lévy processes, given a M-semigroup: the mapping $(\mu(t))_{t\geq 0} \mapsto (\mu_t)_{t\geq 0}$.

The hypergroup $\mathcal{K}$ is embedded into a vector space $\mathbb{H}$, hence inherits a differentiable structure: Note that the action of T_t on $\mathcal{K}$ resp. $\mathbb{H}$ is smooth. In fact, $t \mapsto (T_{\exp tQ}(\kappa))^2 = \exp tQ\kappa^2 \exp tQ^* =: \kappa(t)^2$ is an entire function, and $\mathcal{K} \ni x \mapsto x^{1/2} \in \mathcal{K}$ is holomorphic on the open subset $\mathcal{K}_0 := \mathcal{K} \cap \mathrm{GL}(\mathbb{R}^d)$. If $\kappa \notin \mathrm{GL}(\mathbb{R}^d)$, i.e., if the kernel $N(\kappa) \neq \{0\}$, then $N(\kappa(t)) = \exp(-tQ^*)N(\kappa)$ and $N(\kappa(t))^\perp = \exp(-tQ)N(\kappa)^\perp$, hence the projections onto these subspaces depend analytically on t.

We define particular differential operators:

Definition 5.4 For $f \in C_0^{(1)}(\mathbb{H} \otimes \mathbb{R})$ (i.e. with continuous derivatives in $C_0(\mathbb{H} \otimes \mathbb{R})$) and for $(x,s) \in \mathbb{H} \otimes \mathbb{R}$ we put

$$Xf(x,s) := \left.\frac{\mathrm{d}^+}{\mathrm{d}t}\right|_{t=0} f(T_t(x), s+t) = \lim_{t\searrow 0} \frac{1}{t}(f(T_t(x), s+t) - f(x,s)),$$

$$Pf(x,s) := \left.\frac{\mathrm{d}^+}{\mathrm{d}t}\right|_{t=0} f(x, s+t) = \lim_{t\searrow 0} \frac{1}{t}(f(x, s+t) - f(x,s)),$$

$$Sf(x,s) := \left.\frac{\mathrm{d}^+}{\mathrm{d}t}\right|_{t=0} f(T_t(x), s) = \lim_{t\searrow 0} \frac{1}{t}(f(T_t(x), s) - f(x,s)).$$

For the restriction to $(x,s) \in \Gamma$ we obtain:

Proposition 5.5 *Let* $\lambda \in \mathcal{M}^b(\Gamma)$, $f \in C_0^{(1)}(\mathbb{H} \otimes \mathbb{R})$, $(x,s) \in \Gamma$. *Then*

(a) $Xf(x,s) = \lim_{t\searrow 0} L_{\frac{1}{t}(\varepsilon_{(e,t)}-\varepsilon_{(e,0)})} f(x,s) = \frac{\mathrm{d}}{\mathrm{d}t}|_{t=0} L_{\varepsilon_{(e,t)}} f(x,s)$.
(b) $Pf(x,s) = \lim_{t\searrow 0} R_{\frac{1}{t}(\varepsilon_{(e,t)}-\varepsilon_{(e,0)})} f(x,s) = \frac{\mathrm{d}}{\mathrm{d}t}|_{t=0} R_{\varepsilon_{(e,t)}} f(x,s)$.

Hence, considering $C_0^{(1)}(\mathbb{H}\otimes\mathbb{R})|_\Gamma$ *as a subspace of* $C_0(\Gamma)$, *we obtain*

(c) $R_\lambda Xf(x,s) = XR_\lambda f(x,s)$,
(d) $L_\lambda Pf(x,s) = PL_\lambda f(x,s)$,
(e) $\sup_{(x,s)\in\Gamma} |XR_\lambda f(x,s)| \le \|\lambda\| \sup_{(x,s)\in\Gamma} |Xf(x,s)|$,
(f) $\sup_{(x,s)\in\Gamma} |SR_\lambda f(x,s)| \le \|\lambda\| \sup_{(x,s)\in\Gamma} |Sf((x,s)|$.

⟦(a)–(e) are obvious, only (f) needs a proof:

It is sufficient to prove the assertion for $\lambda = \varepsilon_{(y,u)}$. A simple calculation shows $SR_{(y,u)}f(x,s) = R_{(T_s(y),u)}Sf(x,s)$. Whence

$$\begin{aligned}\sup_{(x,s)\in\Gamma} |SR_{(y,u)}f(x,s)| &= \sup_{(x,s)\in\Gamma} |R_{(T_s(y),u)}Sf(x,s)| \\ &\le \sup_{(y',u)\in\Gamma}\ \sup_{(x,s)\in\Gamma} |R_{(y',u)}Sf(x,s)| \\ &\le \sup_{(y',u)\in\Gamma} \|R_{(y',u)}Sf\|_{C_0(\Gamma)} \le \|Sf\|_{C_0(\Gamma)}.\rrbracket\end{aligned}$$

Proposition 5.6 *Let* $f \in C_0^{(1)}(\mathbb{H}\otimes\mathbb{R})$, $(x,s)\in\mathbb{H}\otimes\mathbb{R}$. *Then*

$$Xf(x,s) = Sf(x,s) + Pf(x,s).$$

⟦By Definition 5.4, $Xf(x,s) = \lim_{t\searrow 0}[\frac{1}{t}(f(T_t(x),s+t) - f(x,s+t)) + \frac{1}{t}(f(x,s+t) - f(x,s))]$.

The limit of the second terms is $Pf(x,s)$, hence also the first terms are convergent, to $S'f(x,s)$ say. Now

$$\begin{aligned}S'f(x,s) = \lim_{t\searrow 0}\Big[&\frac{1}{t}(f(T_t(x),s+t) - f(T_t(x),s)) \\ &+ \frac{1}{t}(f(T_t(x),s) - f(x,s)) - \frac{1}{t}(f(x,s+t) - f(x,s))\Big].\end{aligned}$$

The first and third terms converge to $Pf(x,s)$ and $-Pf(x,s)$ respectively, hence $S'f = Sf$, as asserted.⟧

The differential operators X and P are related by

Proposition 5.7 $(X\widetilde{f})(x,s) = -(\widetilde{Pf})(x,s)$.

$$\begin{aligned}\llbracket\quad X\widetilde{f}(x,s) &= \lim_{t\searrow 0}\frac{1}{t}(\widetilde{f}(T_t(x),s+t) - \widetilde{f}(x,s)) \\ &= \lim_{t\searrow 0}\frac{1}{t}(f(T_{-s-t}T_t(x),-s-t) - f(T_{-s}(x),-s))\end{aligned}$$

$$= \lim_{t\searrow 0} \frac{1}{t}\left(f(T_{-s}(x), -s-t) - f(T_{-s}(x), -s)\right)$$

$$= -(Pf)(T_{-s}(x), -s) = -(\widetilde{Pf})(x,s). \quad]\!]$$

Definition 5.8 We introduce semi-norms on $C_0^{(1)}(\mathbb{H}\otimes\mathbb{R})$:

$$\|f\|_{(0)} := \sup_{(x,s)\in\Gamma} |f(x,s)|, \qquad \|f\|_{(1)} := \sup_{(x,s)\in\Gamma} |Xf(x,s)| = \|Xf\|_{(0)}$$

and $\|f\|_{(2)} := \|Sf\|_{(0)}$. Finally we put $|\!|\!|f|\!|\!| := \sum_{j=0}^{2} \|f\|_{(j)}$.

$\mathcal{B}$ denotes the completion of $C_0^{(1)}(\mathbb{H}\otimes\mathbb{R})$ w.r.t. $|\!|\!|\cdot|\!|\!|$. (Since functions coinciding on Γ are identified, the Banach space $\mathcal{B}$ may be considered as subspace of $C_0(\Gamma)$.)

Proposition 5.9

(a) *$\mathcal{B}$ is dense in $C_0(\Gamma)$ w.r.t. $\|\cdot\|_\infty (= \|\cdot\|_{(0)})$.*
(b) *For all $f\in\mathcal{B}$ there exist Xf, Pf, Sf and belong to $C_0(\Gamma)$.*
(c) *For all $\lambda\in\mathcal{M}^b(\Gamma)$, for all $f\in\mathcal{B}$ we have $|\!|\!|R_\lambda f|\!|\!| \le \|\lambda\|\cdot|\!|\!|f|\!|\!|$.*
(d) *For $\lambda_n, \lambda \in \mathcal{M}_{*+}^{(1)}(\Gamma)$ with $\lambda_n \to \lambda$, we have $R_{\lambda_n} \to R_\lambda$ in the strong operator topology of $\mathcal{B}$. In particular, for a continuous convolution semigroup $(\lambda_t)_{t\ge 0}$ in $\mathcal{M}_{*+}^{(1)}(\Gamma)$ the operators $(R_{\lambda_t})_{t\ge 0}$ may be considered as C_0-contraction semigroup on $(C_0(\Gamma), \|\cdot\|_\infty)$ as well as on $(\mathcal{B}, |\!|\!|\cdot|\!|\!|)$.*

$[\![$(a), (b) are obvious, (c) and (d) are immediate consequences of Proposition 5.5 (e) and (f). In fact, convergence of R_{λ_n} follows observing that X and R_{λ_n} commute and $\|PR_{\lambda_n}f - PR_\lambda f\|_\infty \to 0$ for $f\in C_0^{(1)}(\Gamma)$. The latter is immediately verified considering e.g., $f = \varphi\otimes\psi \in C_0^{(1)}(\mathcal{K})\otimes C_0^{(1)}(\mathbb{R})$.$]\!]$

Definition 5.10 In the following let $(\lambda_t = \mu(t)\otimes\varepsilon_t)_{t\ge 0}$ be a continuous convolution semigroup in $\mathcal{M}_*^1(\Gamma)$ with $\lambda_0 = \varepsilon_{(e,0)}$. Let $(U, D(U))$ resp. $(\overset{*}{U}, D(\overset{*}{U}))$ denote the infinitesimal generators of the C_0-contraction semigroups $(R_{\lambda_t})_{t\ge 0}$ on $C_0(\Gamma)$ and on $\mathcal{B}$ respectively.

Proposition 5.11 *$D(\overset{*}{U})$ is dense in $D(U)$ and in $C_0(\Gamma)$, furthermore, $D(\overset{*}{U})$ is a core for $(U, D(U))$.*

$[\![$In fact, by construction $D(\overset{*}{U}) \subseteq D(U)$ and $D(\overset{*}{U})$ is dense in $\mathcal{B}$ w.r.t. $|\!|\!|\cdot|\!|\!|$. Hence also dense in $C_0(\Gamma)$ w.r.t. $\|\cdot\|_\infty$. Furthermore, $(I - \overset{*}{U})D(\overset{*}{U}) = \mathcal{B}$, hence $(I-U)D(\overset{*}{U})$ is dense in $C_0(\Gamma)$. Whence the assertion.$]\!]$

Remark 5.12 $D(U)$ is left invariant since U is left invariant. But the $|\!|\!|\cdot|\!|\!|$-defining operator S is not left invariant. Hence we can not conclude that $D(\overset{*}{U})$ is left invariant. That is the reason why we have to use more complicated constructions in the sequel. We put $\sigma(t) := \mu(t)\otimes\varepsilon_0$, $p_t^\pm := \varepsilon_e\otimes\varepsilon_{\pm t}$. Hence $\lambda_t = \sigma(t)\star p_t^+$.

Proposition 5.13 *There exists a core $\mathcal{E}$ for $(R_{\lambda_t})_{t\geq 0}$ resp. $(U, D(U))$ such that $\mathcal{E} \subseteq D(U) \cap D(P)$*

Proof Let $f \in D(U)$, $\psi \in \mathcal{D}(\mathbb{R})$. Put $g = g_{f,\psi} : (x, s) \mapsto f(x, s) \cdot \psi(s)$.

Let $\mathcal{E}_0 := \operatorname{span}\{g_{f,\psi} : f \in D(U), \psi \in \mathcal{D}(\mathbb{R})\}$.

1. $\mathcal{E}_0 \subseteq D(U)$.

In fact, we prove for $g := g_{f,\psi} : Ug(x, s) = Uf(x, s) \cdot \psi(s) + f(x, s) \cdot \psi'(s)$:

$$\begin{aligned}
[\![\quad & \frac{1}{t}\int_{\mathcal{K}} g(x \star T_s(y), s+t) - g(x, s)\,\mathrm{d}\mu(t)(y) \\
&= \frac{1}{t}\int_{\mathcal{K}} f(x \star T_s(y), s+t)\cdot\psi(s+t) - f(x, s)\cdot\psi(s)\,\mathrm{d}\mu(t)(y) \\
&= \left[\frac{1}{t}\int_{\mathcal{K}} f(x \star T_s(y), s+t) - f(x, s)\,\mathrm{d}\mu(t)(y)\right]\cdot\psi(s+t) \\
&\quad + \int_{\mathcal{K}} f(x, s)\,\mathrm{d}\mu(t)(y)\cdot\left[\frac{1}{t}(\psi(s+t) - \psi(s))\right] \\
&\xrightarrow{t\to 0} Uf(x, s)\cdot\psi(s) + \psi'(s)\cdot f(x, s). \quad]\!]
\end{aligned}$$

Convergence is uniform in (x, s), since ψ and ψ' have compact supports and $Uf \in C_0(\Gamma)$.

2. $\mathcal{E}_0$ is dense in $C_0(\Gamma)$. In fact, let L_n be compact intervals, $L_n \nearrow \mathbb{R}$, e.g., $L_n = [-k_n, k_n]$ with $k_n \nearrow \infty$. Let $\psi_n \in \mathcal{D}(\mathbb{R})$, $1_{L_n} \leq \psi_n \leq 1_{L_{n+1}}$. Then $f(x, s) \cdot \psi_n(s) \to f(x, s)$ uniformly in $(x, s) \in \Gamma$ (since $f \in C_0(\Gamma)$).

3. $(I - U)\mathcal{E}_0$ is dense in $C_0(\Gamma)$.

[[We show: $\forall \varepsilon > 0\ \forall h \in C_0(\Gamma)$, hence $\forall f \in D(U)$ with $(I - U)f = h$, there exists a $g \in \mathcal{E}_0$ such that $\|(I - U)f - (I - U)g\|_\infty = \|(f - g) - (Uf - Ug)\|_\infty < \varepsilon$. (Note that $(I - U)D(U) = C_0(\mathcal{K})$.)

Let $f \in D(U)$, choose L_n, ψ_n as above, and assume in addition that $\|\psi_n{}'\|_\infty \to 0$. Put $g_n(x, s) := f(x, s)\cdot\psi_n(s)$. Then $(I - U)g_n(x, s) = g_n(x, s) - Uf(x, s)\cdot\psi_n(s) - f(x, s)\cdot\psi_n'(s)$. Therefore, $|(I - U)f(x, s) - (I - U)g_n(x, s)| \leq \|f - g_n\|_{(0)} + |Uf(x, s)| \cdot |1 - \psi_n(s)| + \|f\|_{(0)} \cdot \|\psi_n'\|_\infty \to 0$. Convergence is again uniform in (x, s) since $Uf \in C_0(\Gamma)$.]]

4. The above steps remain true if $\mathcal{E}_0$ is replaced by the subspace

$$\mathcal{E} := \operatorname{span}\{g_{f,\psi} : f \in D(\overset{*}{U}), \psi \in \mathcal{D}(\mathbb{R})\}.$$

(According to Proposition 5.11, $D(\overset{*}{U})$ is a core for $(U, D(U))$.)

5. In that case we have in addition $\mathcal{E} \subseteq D(P)$ and $P\mathcal{E} \subseteq C_0(\Gamma)$.

[[Since $D(\overset{*}{U}) \subseteq \mathcal{B} \subseteq D(P)$ (cf. Proposition 5.9) and $Pg_{f,\psi}(x, s) = Pf(x, s)\cdot\psi(s) + f(x, s)\cdot\psi'(s)$, it follows $Pg_{f,\psi} \in C_0(\Gamma)$.]] □

Note that in contrast to $\mathcal{E}_0$, the core $\mathcal{E}$ is not left invariant but the core $D(U) \cap D(P)$ is:

Proposition 5.14 *$D(U) \cap D(P)$ is a left invariant core for U, P and W: For $f \in D(U) \cap D(P)$ we have:*

$$Uf = Wf + Pf,$$

where $Wf(x,s) = \lim_{t \searrow 0} \frac{1}{t}(R_{\mu(t)\otimes\varepsilon_0} - I)f(x,s) =: \frac{\mathrm{d}^+}{\mathrm{d}t}|_{t=0} R_{\sigma(t)} f(x,s)$.

Proof The first assertion follows since the core $\mathcal{E}$ is contained in $D(U) \cap D(P)$ according to Proposition 5.13. Furthermore, $D(U) \cap D(P)$ is obviously left invariant, since U and P are left invariant. We have

$$\begin{aligned} Uf(x,s) &= \frac{\mathrm{d}^+}{\mathrm{d}t}\bigg|_{t=0} R_{\lambda_t} f(x,s) = \frac{\mathrm{d}^+}{\mathrm{d}t}\bigg|_{t=0} R_{\sigma(t)} R_{p_t^+} f(x,s) \\ &= \lim_{t \searrow 0} \left\{ R_{\sigma(t)} \left[\frac{1}{t}(R_{p_t^+} - I) f(x,s) \right] + \frac{1}{t}(R_{\sigma(t)} - I) f(x,s) \right\}. \end{aligned}$$

The first term converges to $Pf(x,s)$ by assumption, hence also the second term is convergent, i.e., $\frac{\mathrm{d}^+}{\mathrm{d}t}|_{t=0} R_{\sigma(t)} f(x,s) =: Wf(x,s)$ exists, and therefore $Uf = Wf + Pf$ for $f \in D(U) \cap D(P)$ with

$$Wf(x,s) = \frac{\mathrm{d}^+}{\mathrm{d}t}\bigg|_{t=0} R_{\sigma(t)} f(x,s) = \lim_{t \searrow 0} \frac{1}{t} \int_{\mathcal{K}} f(x \star T_s(y), s) - f(x,s)\, \mathrm{d}\mu(t)(y)$$

as asserted. □

Definition 5.15 Put $\Lambda := \{\mathcal{K} \ni x \mapsto f(x,0) : f \in D(U) \cap D(P)\}$, and $\overset{\bullet}{f}: x \mapsto f(x,0)$.

Proposition 5.16 *Λ is $\|\cdot\|_\infty$-dense in $C_0(\mathcal{K})$, left invariant (and also right invariant, as $\mathcal{K}$ is Abelian).*

⟦$D(U) \cap D(P)$ is a left invariant subspace of $C_0(\Gamma)$. In other words, $L_{(y,u)}(D(U) \cap D(P)) \subseteq (D(U) \cap D(P))$ $\forall (y,u) \in \Gamma$. Considering $u = 0$ we obtain $\overset{\bullet}{L}_y (\Lambda) \subseteq \Lambda$ $\forall y \in \mathcal{K}$. Λ is dense in $C_0(\mathcal{K})$ since $D(U) \cap D(P)$ is dense in $C_0(\Gamma)$.⟧

Now we are ready to prove the *existence of a background driving Lévy process:*

Proposition 5.17 *As introduced afore, we write $\overset{\bullet}{f}$ for the restriction of f to $\{(y,0) : y \in \mathcal{K}\} \equiv \mathcal{K}$. With this notation we have:*

$$\Lambda \ni \overset{\bullet}{f} \mapsto Wf(\cdot, 0) =: V\overset{\bullet}{f}$$

is a left invariant operator $\Lambda \to C_0(\mathcal{K})$. V is dissipative (by construction) and has a unique extension to the generator of a semigroup of convolution operators $(R_{\mu_t})_{t\geq 0}$ for a continuous convolution semigroup $(\mu_t)_{t\geq 0} \subseteq \mathcal{M}_+^{(1)}(\mathcal{K})$. In particular, Λ is a core for $(\mu_t)_{t\geq 0}$.

⟦Λ is dense in $C_0(\mathcal{K})$ and left invariant. Since $\mathcal{K}$ is Abelian, Λ is (trivially) right invariant. By construction, V is dissipative and left-invariant, whence, according to Theorem 2.9 (a), the existence of $(\mu_t)_{t\geq 0} \subseteq \mathcal{M}^{(1)}(\mathcal{K})$ with generator V follows.

Positivity is proved as follows: According to Proposition 5.14, $V = \lim_{t\searrow 0} V_t$ where $V_t = \frac{1}{t}(R_{\mu(t)} - I)$ and $\mu(t) \in \mathcal{M}^1(\mathcal{K})$. Hence $R_{\mu_s} = \lim_{t\searrow 0} \exp s \cdot V_t$, thus $\mu_s = \lim_{t\searrow 0} \exp s\frac{1}{t}(\mu(t) - \varepsilon_e) \in \mathcal{M}_+^{(1)}(\mathcal{K})\ \forall s \geq 0$.⟧

Proposition 5.18 *Let $(\mu_t)_{t\geq 0} \subseteq \mathcal{M}_+^{(1)}(\mathcal{K})$, W and V be defined as in Proposition 5.17. Let $(\sigma_t := \mu_t \otimes \varepsilon_0)_{t\geq 0} \subseteq \mathcal{M}_*^{(1)}(\Gamma)$ denote the corresponding continuous convolution semigroup, concentrated on $\mathcal{K} \otimes \{0\} \cong \mathcal{K}$ and put as before $(p_t^{\pm} := \varepsilon_{(e,\pm t)})_{t\in\mathbb{R}_+}$ (cf. Remark 5.12). Then*

(a) *The closures of W and $\pm P$ are the generators of $(R_{\sigma_t})_{t\geq 0}$ and $(R_{p_t^{\pm}})_{t\geq 0}$ respectively.*
(b) *Furthermore, $W = \frac{\mathrm{d}^+}{\mathrm{d}t}|_{t=0} R_{\sigma(t)} = \frac{\mathrm{d}^+}{\mathrm{d}t}|_{t=0} R_{\sigma_t}$.*
(c) *Hence in particular, $\frac{\mathrm{d}^+}{\mathrm{d}t}|_{t=0} \overset{\bullet}{R}_{\mu(t)} = \frac{\mathrm{d}^+}{\mathrm{d}t}|_{t=0} \overset{\bullet}{R}_{\mu_t}$.*

Proof (a) See Proposition 5.14; furthermore, (b) $\Rightarrow$ (c).

To prove (b) note that for all $(x, s) \in \Gamma$ we have:

$$
\begin{aligned}
Wf(x,s) &\overset{\text{Prop. 5.14}}{=} \left.\frac{\mathrm{d}^+}{\mathrm{d}t}\right|_{t=0} R_{\sigma(t)} f(x,s) \\
&= \lim_{t\searrow 0} \frac{1}{t} \int_{\mathcal{K}} f(x \star T_s(y), s) - f(x,s)\, \mathrm{d}\mu(t)(y) \\
&= \lim_{t\searrow 0} \frac{1}{t} \int_{\mathcal{K}} (L_{(e,s)} f)(T_{-s}(x) \star y, 0) - (L_{(e,s)} f)(T_{-s}(x), 0)\, \mathrm{d}\mu(t)(y) \\
&\overset{\text{Prop. 5.17}}{=} V \overset{\bullet}{g}\, (T_{-s}(x)) \quad (\text{with } g := L_{(e,s)} f) \\
&= \left.\frac{\mathrm{d}^+}{\mathrm{d}t}\right|_{t=0} \overset{\bullet}{R}_{\mu_t} \overset{\bullet}{g}\, (T_{-s}(x)) = \left.\frac{\mathrm{d}^+}{\mathrm{d}t}\right|_{t=0} R_{\sigma_t} g(T_{-s}(x), 0) \\
&= \left.\frac{\mathrm{d}^+}{\mathrm{d}t}\right|_{t=0} R_{\sigma_t} f(x,s).
\end{aligned}
$$

□

In view of Proposition 5.14, application of the Lie-Trotter formula (Proposition 3.5 (a)) to the decomposition $U = W + P$ yields

Proposition 5.19 *With the notations introduced above we obtain the Lie Trotter formulas $(LT1)$ and $(LT2)$ (cf. (1.1)).*

⟦The Lie-Trotter formula (Proposition 3.5 (a), (c) resp. (3.2)), applied to $U = W + P$, yields $\lambda_t = \lim_{n\to\infty}(\sigma_{t/n} \star p_{t/n}^{+})^n$. Considering the projection to the $\mathcal{K}$-component we obtain $(LT1)$. $(LT2)$ follows in view of Proposition 5.18 (b) by R. Chernoff's product formula, Proposition 3.5 (b2), applied to the function $t \mapsto \lambda(t) := \mu(t)$.$\rrbracket$

In Proposition 5.17 we have proved $\mu_t \in \mathcal{M}_+^{(1)}(\mathcal{K})$. Now we are ready to prove

Proposition 5.20 $\mu_t \in \mathcal{M}^1(\mathcal{K})$ *for all* $t \geq 0$.

$\llbracket$Assume $\|\mu_t\| < 1$ for some $t > 0$. Then, as μ_t are positive, $\|\mu_t\| = e^{-ct}$ for some $c > 0$. Therefore, in $(LT1)$ the right hand side has norm $\leq e^{-ct}$. A contradiction to the assumption $\mu(t) \in \mathcal{M}^1(\mathcal{K})$.$\rrbracket$

We have proved that for any M-semigroup $(\mu(t))_{t\geq 0} \subseteq \mathcal{M}^1(\mathcal{K})$ there exists a continuous convolution semigroup $(\mu_t)_{t\geq 0} \subseteq \mathcal{M}^1(\mathcal{K})$, the background driving Lévy process, such that $(LT1)$ holds true. In fact, the following results prove uniqueness of $(\mu_t)_{t\geq 0}$ and bijectivity of the mapping $(\mu_t)_{t\geq 0} \leftrightarrow (\mu(t))_{t\geq 0}$. (Bijectivity is a consequence of the inverse Lie-Trotter formula $(LT2)$.)

The existence of M-semigroups given a continuous convolution semigroup: the mapping $(\mu_t)_{t\geq 0} \mapsto (\mu(t))_{t\geq 0}$.

First we show

Theorem 5.21 *Let* $(\mu_t)_{t\geq 0}$ *be a continuous convolution semigroup in* $\mathcal{M}^1(\mathcal{K})$. *Then there exists a M-semigroup* $(\mu(t))_{t\geq 0} \subseteq \mathcal{M}^1(\mathcal{K})$ *such that the Lie Trotter formulas* $(LT1)$ *and* $(LT2)$ (*cf.* (1.1)) *hold.*

At the first glance it seems obvious to consider as before the operator $W = \frac{\mathrm{d}^+}{\mathrm{d}t}|_{t=0} R_{\mu_t \otimes \varepsilon_0} =: \frac{\mathrm{d}^+}{\mathrm{d}t}|_{t=0} R_{\sigma_t}$ and to apply the Lie-Trotter formula to the representation $U = W + P$ resp. $W = U - P$. But a priori there is no 'natural' common domain for U, W, P. Therefore we have to find a slightly different approach. This will be done in the subsequent steps, formulated as propositions.

Let $(\mu_t)_{t\geq 0} \in \mathcal{M}^1(\mathcal{K})$ be given, define $(\sigma_t := \mu_t \otimes \varepsilon_0)_{t\geq 0} \subseteq \mathcal{M}^1_*(\Gamma)$, put for $t > 0$, $W_t := \frac{1}{t}(R_{\sigma_t} - I)$, $V_t := \frac{1}{t}(\overset{\bullet}{R}_{\mu_t} - I)$ (acting on $C_0(\Gamma)$ and $C_0(\mathcal{K})$ respectively). Furthermore, let $(W, D(W))$ and $(V, D(V))$ be the generators of the corresponding contraction semigroups (R_{σ_t}) and $(\overset{\bullet}{R}_{\mu_t})$.

Let $\mathcal{A} \subseteq D(V)$ denote a core for $(\mu_t)_{t\geq 0}$ with the following properties:

(1) $\mathcal{A}$ is left invariant (and right invariant, as $\mathcal{K}$ is Abelian).
(2) $T_s(\mathcal{A}) \subseteq \mathcal{A}$ for all automorphisms T_s.

[Such cores exist for $\mathcal{K}$, e.g., $\mathcal{A} = (L^1_c(\widehat{\mathcal{K}}))^\vee$, the space of analytic vectors, as mentioned in Corollary 2.12 and Remark 2.13.]

Define $\mathbb{D} := \mathcal{A} \otimes \mathcal{D}(\mathbb{R}) \subseteq C_0(\Gamma)$. Then we have:

(i) $\mathbb{D} \subseteq D(W)$

$\llbracket$Let $f := \varphi \otimes \psi \in \mathbb{D}$. Then, with $\gamma := \varphi \circ T_s \in \mathcal{A}$ $W_t f(x,s) = \frac{1}{t}(\int_{\mathcal{K}} \varphi(x \star T_s(y)) - \varphi(x)\,\mathrm{d}\mu_t(y)) \cdot \psi(s) = (V_t\gamma)(T_{-s}(x)) \cdot \psi(s) \xrightarrow{t\to 0} (V\gamma)(T_{-s}(x)) \cdot \psi(s)$. Thus, $(x,s) \mapsto Wf(x,s) := \lim_{t\searrow 0} W_t f(x,s) \in C_0(\Gamma)$, hence $f \in D(W)$.$\rrbracket$

(ii) $\mathbb{D}$ is left invariant and dense in $C_0(\Gamma)$.

⟦Obviously $\mathbb{D}$ is dense in $C_0(\Gamma)$. To prove invariance we consider $L_{(y,t)}(\varphi \otimes \psi)(x,s) = (\varphi \otimes \psi)(y \star T_t(x), s+t) = (\varphi \circ T_t)(T_{-t}(y) \star x) \cdot \psi_t(s) = \overset{\bullet}{L}_{(T_{-t}(y))} (\varphi \circ T_t)(x) \cdot \psi_t(s) =: g(x) \cdot \psi_t(s)$ with $g = \overset{\bullet}{L}_{(T_{-t}(y))} (\varphi \circ T_t) \in \mathcal{A}$ and $\psi_t(\cdot) := \psi(t + \cdot) \in \mathcal{D}(\mathbb{R})$.⟧

Proposition 5.22 *Let* $f \in \mathbb{D}$, $(z,u) \in \Gamma$. *Then* $R_{(z,u)} f \in D(W)$.

Proof In fact, by definition

$$\begin{aligned} &W_t R_{(z,u)}(\varphi \otimes \psi)(x,s) \\ &\quad = \frac{1}{t} \int (\varphi(x \star T_s(z) \star T_{s+u}(y)) - \varphi(x \star T_s(z))\, \mathrm{d}\mu_t(y) \cdot \psi_u(s) \\ &\quad = V_t(\varphi \circ T_{s+u})(T_{-(s+u)}(x) \star T_u(z)) \cdot \psi_u(s) \\ &\quad =: \overset{\bullet}{R}_z\, ((V_t \varphi_{s,u}) \circ T_{-u})(T_{-s}(x)) \cdot \psi_u(s). \end{aligned}$$

Hence, with $\varphi_{s,u} := \varphi \circ T_{s+u}$, ψ_u as above,

$$\begin{aligned} &W_t R_{(z,u)}(\varphi \otimes \psi)(x,s) \xrightarrow{t \to 0} \overset{\bullet}{R}_z\, ((V \varphi_{s,u}) \circ T_{-u})(T_{-s}(x)) \cdot \psi_u(s) \\ &\quad = V(\varphi \circ T_{s+u})(T_{-(s+u)}(x) \star T_u(z)) \cdot \psi(s+u) \\ &\quad = W(R_{(z,u)}(\varphi \otimes \psi))(x,s). \end{aligned}$$

Convergence is again uniform on Γ, hence $R_{(z,u)} f \in D(W)$. ☐

Definition 5.23 Let $\widetilde{\mathbb{D}} := \operatorname{span}\{R_{(z,u)} f : (z,u) \in \Gamma,\ f \in \mathbb{D}\}$.

Proposition 5.24 $\widetilde{\mathbb{D}}$ *is dense in* $C_0(\Gamma)$ *and left and right invariant.*

Furthermore, $\widetilde{\mathbb{D}} \subseteq D(W) \cap D(P)$.

W and $\pm P$ are, as limits of convolution operators, left invariant and by construction dissipative. Hence $U = W + P$ shares this property.

Therefore, according to Theorem 2.9 (c), $\widetilde{\mathbb{D}}$ *is a core for P, $U := W + P$, and hence also for $W = U - P$.*

⟦Only $\widetilde{\mathbb{D}} \subseteq D(P)$ needs a proof:

$$\begin{aligned} P R_{(z,u)}(\varphi \otimes \psi)(x,s) &= \lim_{t \searrow 0} R_{\frac{1}{t}(\varepsilon_{(e,t)} - \varepsilon_{(e,0)})} R_{(z,u)}(\varphi \otimes \psi)(x,s) \\ &= \lim_{t \searrow 0} (\varphi(x \star T_s(z))) \cdot \frac{1}{t}(\psi(s+u+t) - \psi(s+u)) \\ &= \varphi(x \star T_s(z)) \cdot \psi'(s+u). \end{aligned}$$

Convergence is uniform since ψ and ψ' have compact supports. Whence the assertion.⟧

Proposition 5.25 *The Lie-Trotter formulas* $(LT1)$ *and* $(LT2)$ *hold.*

⟦Applying the Lie-Trotter formula (cf. Proposition 3.5 (a)) to $U = W + P$ resp. $W = U - P$ yields in view of Proposition 5.24, $\lambda_t = \lim_{n\to\infty}(\sigma_{t/n} * p^+_{t/n})^n$ resp. $\sigma_t = \lim_{n\to\infty}(\lambda_{t/n} * p^-_{t/n})^n$, $t \geq 0$. Projecting to the space component $\mathcal{K}$ yields $(LT1)$ resp. $(LT2)$ (cf. (1.1)).⟧

We have proved, that $(\mu(t))_{t\geq 0} \subseteq \mathcal{M}^{(1)}_+(\mathcal{K})$, $(\lambda_t)_{t\geq 0} \subseteq \mathcal{M}^{(1)}_+(\Gamma)$. Comparing the norms in $(LT1)$ and $(LT2)$ yields again—as in Proposition 5.20—that $\mu(t)$ and hence λ_t are probabilities.

The proof of Theorem 4.2 is complete. □

Appendix

In Theorem 4.2 we supposed $\mathcal{K}$ to be an Abelian hypergroup in order to ensure the existence of a common core for continuous convolution semigroups. In fact, the proof is written for matrix cone hypergroups, however it could be generalized to larger classes of—not necessarily Abelian—hypergroups defined on a state space $\mathcal{K}$ embedded into a space $\mathbb{H}$ endowed with a differentiable structure and $(\mathcal{M}^1(\mathcal{K}), \star)$ admitting one parameter groups of automorphisms. But in the moment no further examples are available. (As mentioned afore, it is quite reasonable to investigate the 'Heisenberg convolutions' on $\mathbb{K} \rtimes \mathbb{R}$ in [55] as first non-commutative examples. This will be done in future.)

If we restrict to the Abelian situation we are able to find an alternative proof for the existence of the mapping $(\mu_t)_{t\geq 0} \mapsto (\mu(t))_{t\geq 0}$.

The alternative proof avoids space-time semigroups and relies heavily on Abelian harmonic analysis, in particular on Fourier transforms, positive and negative definite functions and on the validity of Lévy's continuity theorem for hypergroups ([6], Theorems 4.2.4, 4.2.5). Let in the following $\mathcal{K}$ denote a commutative hypergroup with dual $\widehat{\mathcal{K}}$ such that Lévy's continuity theorem is valid. (For definitions and properties of positive and negative definite functions on hypergroups see e.g. [6], 4.2, 4.4, [24, 54].) Furthermore, we assume the existence of a continuous one parameter group $(T_t) \subseteq \mathrm{Aut}(\mathcal{K})$.

Proposition 6.1 *Let* (μ_t) *be a continuous convolution semigroup. Then there exists a continuous M-semigroup* $(\mu(t))$ *such that* $(LT1)$ *holds.*

Furthermore, let $\widehat{\mu_t} = e^{tL}$, *where* $-L : \widehat{\mathcal{K}} \to \mathbb{R}$ *is strongly negative definite. Then* $\widehat{\mu(t)} = \mathrm{e}^{M(t)}$, $t \geq 0$, *for strongly negative definite functions* $-M(t) : \widehat{\mathcal{K}} \to \mathbb{C}$, *such that*

$$\frac{\mathrm{d}^+}{\mathrm{d}t}\bigg|_{t=0} \widehat{\mu(t)} = \frac{\mathrm{d}^+}{\mathrm{d}t}\bigg|_{t=0} M(t) = \frac{\mathrm{d}^+}{\mathrm{d}t}\bigg|_{t=0} \widehat{\mu_t} = L. \tag{6.1}$$

Proof L is a continuous function and $\mathbb{R} \ni s \mapsto T_s \in \mathrm{Aut}(\mathcal{K})$ is continuous. Define

$$M(t) := \int_0^t L \circ T_s^* \mathrm{d}s = \lim_{n\to\infty} \frac{t}{n} \sum_{k=0}^{n-1} L \circ T^*_{kt/n} =: \lim_{n\to\infty} M_n(t)$$

(where T_s^* denotes the dual automorphism acting on the dual $\widehat{\mathcal{K}}$, $T_s^*(\varphi) := \varphi \circ T_s$ for $\varphi \in \widehat{\mathcal{K}}$).

Obviously, $-M_n(t)$, $t \geq 0$, are continuous strongly negative definite functions with $(\pi_t^{(n)} := \bigstar_{k=0}^{n-1} \mu_{k,t}^{(n)})_{t\geq 0}$ as corresponding continuous convolution semigroups, where $\mu_{k,t}^{(n)} := T_{kt/n}(\mu_{t/n})$ and $\widehat{\pi_t^{(n)}} = e^{M_n(t)}$. Moreover, we have $e^{M_n(t)} \overset{n\to\infty}{\longrightarrow} e^{M(t)}$ (for all $t \geq 0$), and the limit is continuous. Therefore, according to Lévy's continuity theorem for hypergroups (cf. [6], Theorem 4.2.4), there exist probabilities $\mu(t) \in \mathcal{M}^1(\mathcal{K})$ with $\widehat{\mu(t)} = e^{M(t)}$. And, since by construction, $t \mapsto M(t)$ is continuous, $t \mapsto \widehat{\mu(t)}$ is continuous, hence again according to Lévy's continuity theorem, $t \mapsto \mu(t)$ is weakly continuous. Furthermore, by construction,

$$\mu(t) = \lim_{n\to\infty} \pi_t^{(n)} = \lim_{n\to\infty} \bigstar_{k=0}^{n-1} T_{kt/n}(\mu_{t/n}).$$

I.e., $(LT1)$ holds. And in addition, $\forall s, t \geq 0$,

$$M(s+t) = M(s) + M(t) \circ T_s^*. \tag{6.2}$$

As easily verified, property (6.2) is equivalent to $(\mu(t))_{t\geq 0}$ being a M-semigroup (cf. (4.1)). Furthermore, as $\widehat{\mu(t)} = \mathrm{e}^{M(t)}$, $\widehat{\mu_t} = \mathrm{e}^{tL}$ we obtain:

$$\left.\frac{\mathrm{d}^+}{\mathrm{d}t}\right|_{t=0} \widehat{\mu(t)} = \left.\frac{\mathrm{d}^+}{\mathrm{d}t}\right|_{t=0} \widehat{\mu_t} = L.$$

In addition, by definition, $M(t) = \int_0^t L \circ T_s^* \,\mathrm{d}s$, hence $t \mapsto M(t)$ is point-wise differentiable at $t = 0$ with

$$\left.\frac{\mathrm{d}^+}{\mathrm{d}t}\right|_{t=0} M(t) = L.$$

The assertion is proved. ☐

Under additional assumptions, in particular if differentiability of $t \mapsto \widehat{\mu(t)}$ is assumed, a converse result can be proved:

Proposition 6.2 *Let $(\mu(t))$ be a M-semigroup. Assume $\widehat{\mu(t)} = \mathrm{e}^{M(t)}$, with strongly negative definite functions $-M(t)$, $t \geq 0$. Assume furthermore, that $t \mapsto M(t)$ is point-wise differentiable at $t = 0$, with $\frac{\mathrm{d}^+}{\mathrm{d}t}|_{t=0} M(t) = L$, where $-L$ is strongly negative definite and continuous. (Or, equivalently, assume $\frac{\mathrm{d}^+}{\mathrm{d}t}|_{t=0} \widehat{\mu(t)} = L$.)*

Then there exists a continuous convolution semigroup (μ_t) such that $(LT2)$ holds.

Proof $M(\cdot)$ satisfies (6.2). Hence differentiability of $M(\cdot)$ resp. $\widehat{\mu(\cdot)}$ yields $\widehat{\mu(t/n)}^n = e^{t\cdot\frac{n}{t}\cdot M(\frac{t}{n})} \to e^{t\cdot L}$. By Lévy's continuity theorem $(LT2)$ follows: $\mu(t/n)^n \to \mu_t$. ☐

(See also the discussion of differentiability of $\widehat{\mu(\cdot)}$ in the case of M-semigroups on vector spaces in [36], Remark 5 (b).)

Putting things together, in view of the proof of Theorem 4.2, especially Propositions 5.18, 6.1, and 6.2 it follows:

Theorem 6.3 *Let again* $\mathcal{K}$ *be a matrix cone hypergroup. Then for any M-semigroup on* $\mathcal{K}$ *with Fourier transform* $\widehat{\mu(t)}$ *there exist strongly negative definite continuous functions* $-M(\cdot)$ *such that* $\widehat{\mu(t)} = e^{M(t)}$, $t \geq 0$, *and* $t \mapsto M(t)$ *is differentiable at* $t = 0$ *with* $\frac{\mathrm{d}^+}{\mathrm{d}t}|_{t=0} M(t) = L$. *I.e.,* L*, the logarithm of the Fourier transform of the background driving Lévy process, is the derivative of* $t \mapsto M(t)$ *at* $t = 0$. *And conversely, given* L*, then* $t \mapsto M(t) := \int_0^t L \circ T_s^* \,\mathrm{d}s$ *satisfies* $\widehat{\mu(t)} = e^{M(t)}$ *as well as* (6.2) *and* (6.1).

⟦1. In fact, for any continuous convolution semigroup with $\widehat{\mu_t} = \mathrm{e}^{tL}$, $M(t) = \int_0^t L \circ T_s^* \,\mathrm{d}s$ and $\widehat{\mu(t)} = \mathrm{e}^{M(t)}$, we obtain a M-semigroup by Proposition 6.1.

2. And for a given M-semigroup $(\mu(t))$, by Theorem 4.2 there exists a continuous convolution semigroup (μ_t) with $\widehat{\mu_t} = \mathrm{e}^{tL}$, bijectively related by (LT1) and (LT2). Hence by step 1, we obtain the existence of functions $M(\cdot)$ such that (6.2) and (6.1) hold true.⟧

References

1. Becker-Kern, P.: Stable and semistable hemigroups: domains of attraction and self-decomposability. J. Theor. Probab. **16**, 573–598 (2001)
2. Becker-Kern, P.: Random integral representation of operator-semi-self-similar processes with independent increments. Stoch. Process. Appl. **109**, 327–344 (2004)
3. Becker-Kern, P., Hazod, W.: Mehler hemigroups and embedding of discrete skew convolution semigroups on simply connected nilpotent Lie groups. In: Hilgert, J., Hora, A., Kawazoe, T., Nishiyama, K., Voit, M. (eds.) Infinite Dimensional Harmonic Analysis IV. Proceedings, Tokyo, 2007, pp. 32–46. World Scientific, Singapore (2009)
4. Bingham, N.H.: Lévy processes and selfdecomposability in finance. Probab. Math. Stat. **26**, 367–378 (2006)
5. Bogachev, V.I., Röckner, M.: Generalized Mehler semigroups and applications. Probab. Theory Relat. Fields **105**, 193–225 (1996)
6. Bloom, W., Heyer, H.: Harmonic Analysis of Probability Measures on Hypergroups. Walter de Gruyter, Berlin (1995)
7. Chernoff, P.: Note on product formulas for operator semigroups. J. Funct. Anal. **2**, 238–242 (1968)
8. Chernoff, P.: Product formulas, nonlinear semigroups and addition of unbounded operators. Mem. AMS **140** (1974)
9. Drisch, Th., Hazod, W.: Analytische Vektoren von Faltungshalbgruppen I. Math. Z. **172**, 1–28 (1980)
10. Dawson, D.A., Li, Z., Schmuland, B., Sun, W.: Generalized Mehler semigroups and catalytic branching processes with immigration. Potential Anal. **21**, 75–97 (2004)
11. Duflo, M.: Représentations de semi-groupes de mesures sur un groupe localement compact. Ann. Inst. Fourier **28**(3), 225–249 (1978)
12. Duflo, M.: Semigroups of complex measures on a locally compact group. In: Non-Commutative Harmonic Analysis. Lecture Notes Math., vol. 466, pp. 56–64 (1975)
13. Faraut, J.: Semigroupe de mesures complexes et calcul symbolique sur les générateurs infinitésimaux de semigroupes d'opérateurs. Ann. Inst. Fourier **20**, 235–301 (1970)
14. Faraut, J.: Semigroupe de mesures complexes sur un espace homogène et distributions dissipatives. Symp. Math. **XXI**, 257–265 (1977)
15. Faraut, J., Harzallah, K.: Semigroupes d'opérateurs invariants et opérateurs dissipatifs invariants. Ann. Math. Inst. Fourier **22**, 147–164 (1972)
16. Goldstein, J.A.: Semigroups of Linear Operators and Applications. Oxford University Press, Oxford (1985)

17. Hazod, W.: Stetige Faltungshalbgruppen von Wahrscheinlichkeitsmaßen und erzeugende Distributionen. Lecture Notes Math., vol. 595 (1977)
18. Hazod, W.: On some convolution semi- and hemigroups appearing as limit distributions of normalized products of group-valued random variables. In: Heyer, H., Marion, J. (eds.) Analysis on Infinite-dimensional Lie Groups, Marseille, 1997, pp. 104–121. World Scientific, Singapore (1998)
19. Hazod, W.: On Mehler semigroups, stable hemigroups and selfdecomposability. In: Heyer, H., Hirai, T., Kawazoe, T., Saito, K. (eds.) Infinite Dimensional Harmonic Analysis III. Proceedings, 2003, pp. 83–98. World Scientific, Singapore (2005)
20. Hazod, W.: Probability on matrix-cone hypergroups: limit theorems and structural properties. J. Appl. Anal. **15**, 205–245 (2009)
21. Hazod, W., Siebert, E.: Stable Probability Measures on Euclidean Spaces and on Locally Compact Groups. Structural Properties and Limit Theorems. Mathematics and Its Applications, vol. 531. Kluwer Academic, Norwell (2001)
22. Heyer, H.: Probability Measures on Locally Compact Groups. Springer, Berlin (1977)
23. Heyer, H.: Semi-groupes de convolution sur un groupe localement compact et applications a la théorie des probabilités. In: Ecole d'eté de probabilités de Saint-Flour VII. Lecture Notes Math., vol. 678, pp. 173–136 (1978)
24. Heyer, H.: Positive and negative definite functions on a hypergroup and its dual. In: Hilgert, J., Hora, A., Kawazoe, T., Nishiyama, K., Voit, M. (eds.) Infinite Dimensional Harmonic Analysis IV. Proceedings, Tokyo, 2007, pp. 63–96. World Scientific, Singapore (2009)
25. Hirsch, F.: Opérateurs dissipatifs et codissipatifs invariants par translation sur les groupes localement compacts. Séminaire de théorie de potentiel. 15ième année (1971/1972)
26. Hirsch, F.: Sur les semi-groupes d'opérateurs invariants par translation. C. R. Acad. Sci., Paris **274**, 43–46 (1972)
27. Hirsch, F.: Familles résolvantes, générateurs, cogénérateurs, potentiels. Ann. Inst. Fourier **22**, 89–210 (1972)
28. Hirsch, F., Roth, J.P.: Opérateurs dissipatifs et codissipatifs invariants sur un espace homogène. C. R. Acad. Sci., Paris **274**, 1791–1793 (1972)
29. Hirsch, F., Roth, J.P.: Opérateurs dissipatifs et codissipatifs invariants par translation sur un espace homogène. In: Non Commutative Harmonic Analysis. Lecture Notes Math., vol. 404, pp. 229–245 (1974)
30. Hofmann, K.H., Jurek, Z.: Some analytical semigroups occurring in probability theory. J. Theor. Probab. **9**, 745–763 (1996)
31. Hulanicki, A.: A class of convolution semi-groups of measures on a Lie group. In: Weron, A. (ed.) Probability on Vector Spaces II, Proceedings, Błażejewko, 1979. Lecture Notes Math., vol. 828, pp. 82–101 (1980)
32. Jeanblanc, M., Pitman, J., Yor, M.: Selfsimilar processes with independent increments associated with Lévy Bessel processes. Stoch. Process. Appl. **100**, 223–231 (2002)
33. Jurek, Z.: An integral representation of operator-selfdecomposable random variables. Bull. Acad. Pol. Sci. **30**, 385–393 (1982)
34. Jurek, Z.: Self-decomposability: an exception or a rule? Ann. Univ. Curie-Sklodowska, Lublin, Sect. A **51**, 93–107 (1997)
35. Jurek, Z.: On relations between Urbanik and Mehler semigroups. Manuscript (2008)
36. Jurek, Z.: Measure valued cocycles from my papers in 1986 and 1983 and Mehler semigroups. Manuscript (2008)
37. Jurek, Z., Mason, D.: Operator Limit Distributions in Probability Theory. Wiley, New York (1993)
38. Jurek, Z., Vervaat, W.: An integral representation for self-decomposable Banach space valued random variables. Z. Wahrscheinlichkeitstheor. Verw. Geb. **62**, 247–262 (1983)
39. Kato, T.: Perturbation Theory of Linear Operators. Springer, Berlin (1966)
40. Kunita, H.: Stochastic processes with independent increments in a Lie group and their self similar properties. In: Stochastic Differential and Difference Equations. Proceedings, Györ, 1996. Progress Syst. Control Theory, vol. 23, pp. 183–201 (1997)
41. Li, Z.: Skew convolution semigroups and related immigration processes. Theory Probab. Appl. **46**, 274–296 (2002)
42. Lumer, G., Philipps, R.S.: Dissipative operators in a Banach space. Pac. J. Math. **11**, 679–698 (1961)
43. Maejima, M., Sato, K.I.: Semi Lévy processes, self similar additive processes and semi stationary Ornstein-Uhlenbeck type processes. J. Math. Kyoto Univ. **43**, 609–639 (2003)
44. Menges, S.: Stetige Faltungshalbgruppen und Grenzwertsätze auf Hypergruppen. Dissertation, Universität Dortmund (2003). http://eldorado.tu-dortmund.de/handle/2003/2313

45. Nakazato, H.: On left invariant dissipative operators. Arch. Math. **45**, 458–462 (1985)
46. Roth, J.P.: Opérateurs dissipatifs et semi-groupes dans les espaces des fonctions continues. Ann. Inst. Fourier **26**(4), 1–97 (1973)
47. Roth, J.P.: Sur les semi-groupes à contraction invariants sur un espace homogène C.R. Acad. Sci. Paris, Sér. A **227**, 1091–1094 (1973)
48. Rösler, M.: Bessel convolutions on matrix cones. Compos. Math. **143**, 749–779 (2007)
49. Sato, K.: Stochastic integrals of additive processes and application to semi-Lévy processes. Osaka J. Math. **41**, 211–236 (2004)
50. Sato, K., Yamazato, M.: Operator-selfdecomposable distributions as limit distributions of processes of Ornstein–Uhlenbeck type. Nagoya Math. J. **97**, 71–94 (1984)
51. Schmuland, B., Sun, W: On the equation $\mu_{s+t} = \mu_s \star T_s(\mu_t)$. Stat. Probab. Lett. **52**, 183–188 (2001)
52. Urbanik, K.: Lévy's probability measures on Euclidean spaces. Stud. Math. **44**, 119–148 (1972)
53. Voit, M.: Bessel convolutions on matrix cones: algebraic properties and random walks. J. Theor. Probab. **22**, 741–771 (2009)
54. Voit, M.: Positive and negative definite functions on the dual space of a commutative hypergroup. Analysis **9**, 371–387 (1989)
55. Voit, M.: Multidimensional Heisenberg convolutions and product formulas for multivariate Laguerre polynomials. Manuscript (2010). Preprints TU Dortmund 2010-08: tu-dortmund/MathPreprints
56. Zeuner, H.M.: Complex Lévy measures. In: Heyer, H. (ed.) Probability Measures on Groups VII. Proceedings, Oberwolfach, 1983. Lecture Notes Math., vol. 1064, pp. 471–480 (1984)
57. Zeuner, H.M.: The Lévy Khintchine-formula for dissipative distributions. Math. Ann. **274**, 273–282 (1986)