CONTINUOUS CONVOLUTION HEMIGROUPS INTEGRATING A SUB-MULTIPLICATIVE FUNCTION

WILFRIED HAZOD

Abstract. Unifying and generalizing previous investigations for vector spaces and for locally compact groups, E. Siebert obtained the following remarkable result: A Lévy process on a completely metrizable topological group $\mathbb{G}$, resp. a continuous convolution semigroup of probabilities satisfies a moment condition $\int f \mathrm{d}\mu_t < \infty$ for some sub-multiplicative function $f > 0$ if and only if the jump measure of the process resp. the Lévy measure η of the continuous convolution semigroup satisfies $\int_{\complement U} f \mathrm{d}\eta < \infty$ for some neighbourhood U of the unit e. Here we generalize this result to additive processes resp. convolution *hemi*groups $(\mu_{s,t})_{s \leq t}$ on (second countable) locally compact groups.

Introduction

A probability ν on a normed vector space $(\mathbb{V}, ||\cdot||)$ possesses a $k-$th moment, if $\int ||x||^k \mathrm{d}\nu < \infty$, equivalently, if $f : x \to (1 + ||x||)^k$ is $\nu-$integrable. f is continuous, sub-multiplicative, symmetric and satisfies $f(0) = 1$. Hence moment conditions are integrability conditions for (particular) sub-multiplicative functions.

For investigations in limit theorems on more general structures, in particular on locally compact groups, investigations of integrability of sub-multiplicative functions provide interesting tools. In [27], Theorem 1, [28], Theorem 5, E. Siebert obtained characterizations of integrability of such f for continuous convolution semigroups resp. for Lévy processes, in terms of the behaviour of the Lévy measures, resp. the jump-measures of the processes: [27] is based on analytical methods whereas in [28] the emphasis is laid on the behaviour of the processes. In fact, a partial key result (for processes with uniformly bounded jumps resp. for Lévy measures with uniformly bounded supports), [28], Theorem 4, is proved for additive processes resp. for convolution hemigroups. Whereas the general characterization of integrability of sub-multiplicative f (relying on [28], Theorem 5,) is proved there only for continuous convolution semigroups resp. for Lévy processes.

Date: 11.2.2010.

For vector spaces this characterization was proved almost simultaneously by Z. Jurek and S. Smalara [17] and A. de Acosta [1]. For partial results on groups see e.g., [15], [21] and the references in [27, 28]. In [28] E. Siebert proved this result for completely metrizable topological groups, unifying previous investigations for vector spaces and groups.

These characterizations were generalized for special sub-mutiplicative functions f (*logarithmic moments*) and for particular *hemi*groups resp. additive processes arising in connection with self-decomposability resp. (generalized) Ornstein-Uhlenbeck processes: For vector spaces see e.g., the monograph [16], 3.6.6 for homogeneous groups see e.g. [6], §2.14 VII, [8]. ('Logarithmic moments' are defined by the sub-multiplicative functions $f : x \mapsto 1 + \log(1 + ||x||) \approx \log^+ ||x||$.)

Hemigroups resp. additive processes turned out to be essential for investigations in various applications. The background for hemigroups on locally compact groups is found e.g., in [29], [10], [11], [12] and the references mentioned there; see also [3].

E. Siebert's proofs ([28], Theorem 5, resp. [27], Theorem 1) rely on a splitting of the underlying Lévy measure η of the continuous convolution semigroup $(\mu_t)_{t\geq 0}$ (resp. the jump-measure of the underlying process) into a part η_1 with bounded support V and a bounded measure η_2 concentrated on $\complement V$. Hence we obtain two continuous convolution semigroups $\left(\mu_t^{(i)}\right)_{t\geq 0}, i = 1, 2$: For the first any f is integrable, the second one is a Poisson semigroup with generator $\gamma = c \cdot (\rho - \varepsilon_e) =: \eta_2 - ||\eta_2|| \cdot \varepsilon_e$, and the underlying continuous convolution semigroup $(\mu_t)_{t\geq 0}$ is represented by a perturbation series in terms of $\left(\mu_t^{(1)}\right)_{t\geq 0}$ and γ. This technique allows to reduce the investigations to the Poisson part, and we obtain ([27], [28]): f is integrable w.r.t. the underlying continuous convolution semigroup iff f is integrable w.r.t. η_2, the bounded part of the Lévy measure.

Here, in Theorem 4.3, we generalize Siebert's results to (Lipschitz-continuous) convolution *hemigroups* on locally compact groups. As mentioned above, the original proofs rely on a representation by perturbation series. Therefore, we start in Section 1 with perturbation series for *operator hemigroups* (also called generalized semigroups or evolution families) to provide the tools for the next sections. Then, applying this result to convolution operators and following (and generalizing) the proofs in [28] resp. [27], we obtain a version of Siebert's characterization in the general situation (Theorem 4.3). At the first glance, a slightly weaker version, since an additional technical condition (2.1) resp. (4.4) is needed. This condition is however always satisfied for continuous convolution semigroups.

In the appendix we sketch briefly some applications and examples.

1. PERTURBATION SERIES REPRESENTATIONS FOR HEMIGROUPS OF OPERATORS

Definition 1.1. *Let* $\mathbb{B}$ *be a separable Banach space, and* $\mathcal{B}(\mathbb{B})$ *the Banach space of bounded operators. A family* $\{U_{t,t+s}\}_{0\le t\le t+s\le T} \subseteq \mathcal{B}(\mathbb{B})$, $(T \le \infty)$ *is called continuous hemigroup of operators if* $(s,t)\mapsto U_{t,t+s}$ *is continuous w.r.t. the strong operator topology,* $U_{s,s} = I$ *for all* s, *and* $U_{s,r}U_{r,t} = U_{s,t}$ *for all* $s \le r \le t$, *and finally* $||U_{t,t+s}|| \le M\mathrm{e}^{\beta s}$ *for all* $t, s \ge 0$, *for some* $M \ge 1$ *and* $\beta \ge 0$.

To simplify notations, here we shall throughout restrict to the case $M = 1$ *and frequently also* $\beta = 0$, *i.e., we restrict to contractions.*

Hemigroups of operators were investigated under different notations, e.g., *evolution families* or *evolution operators* (cf. [19], [20], [9], [13]) or *semi-groupes generaliseés* ([23]), etc. In view of the applications to distributions of additive processes we prefer the expression *operator hemigroups* (cf. [11]) in analogy to the standard notations in probability theory.

Theorem 1.2. ***a)*** *Let* $\{U_{s,t}\}_{0\le s\le t}$ *be a continuous hemigroup of contractions. Let* $\mathbb{R} \ni t \mapsto C(t) \in \mathcal{B}(\mathbb{B})$ *be a measurable mapping, uniformly bounded,* $||C(t)|| \le \beta$ *for all* $t \ge 0$. *Then*

$$V_{t,t+s} := \sum_{k\ge 0} V^{(k)}_{t,t+s} \quad \text{with} \quad V^{(0)}_{t,t+s} := U_{t,t+s},$$

$$V^{(k+1)}_{t,t+s} := \int_0^s V^{(0)}_{t,t+u} C(t+u) V^{(k)}_{t+u,t+s}\mathrm{d}u$$

defines a continuous operator hemigroup satisfying a growth condition $||V_{t,t+s}|| \le \mathrm{e}^{\beta s}$ *for all* $t, s \ge 0$.

b) *If* $s \mapsto U_{t,t+s}$ *is a.e. differentiable with* $\frac{\partial^+}{\partial s} U_{t,t+s}|_{s=0}(x) =: A(t)(x)$ *for* $x \in D(A(t))$, *and if* $\mathbb{D} := \bigcap\limits_{t\ge 0} D(A(t))$ *is dense, then for all* $x \in \mathbb{D}$ $s \mapsto V_{t,t+s}(x)$ *is differentiable a.e. with* $\frac{\partial^+}{\partial s} V_{t,t+s}(x)|_{s=0} = A(t)x + C(t)x$, *resp. in integrated form:* $V_{t,t+s}(x) = \int_0^s V_{t,t+u}\left(A(u) + C(u)\right)(x)\mathrm{d}u$

c) *In particular, let* $C(t) = c(t)(S(t) - I)$ *with contractions* $S(\cdot)$, $0 \le c(\cdot) \le \beta$, *where* $t \mapsto c(t)$ *and* $t \mapsto S(t)$ *are measurable. Then we* *obtain representations*

$$\begin{aligned} V_{t,t+s} &= \mathrm{e}^{-\beta s}\sum_{k\geq 0} W^{(k)}_{t,t+s}, \quad \textit{with} \quad ||W^{(k)}_{t,t+s}|| \leq \frac{\beta^k s^k}{k!} \qquad (1.1)\\ W^{(0)}_{t,t+s} &:= U_{t,t+s}, \quad W^{(k+1)}_{t,t+s} := \int_0^s W^{(0)}_{t,t+u}\widetilde{C}(t+u)W^{(k)}_{t+u,t+s}\mathrm{d}u,\\ \textit{where}\quad \widetilde{C}(\tau) &= C(\tau)+\beta\cdot I = c(\tau)S(\tau)+(\beta-c(\tau))\cdot I \end{aligned}$$

Hence $||V_{t,t+s}|| \leq 1, 0 \leq t \leq t+s \leq T$. *Alternatively,*

$$V_{t,t+s} = \mathrm{e}^{-\beta s}\sum_{k\geq 0}\frac{s^k\beta^k}{k!}\widetilde{W}^{(k)}_{t,t+s} \quad \textit{with} \quad ||\widetilde{W}^{(0)}_{t,t+s}|| \leq 1, \widetilde{W}^{(k)}_{t,t+s} := \frac{k!}{s^k\beta^k}W^{(k)}_{t,t+s}$$

Proof. Consider the Banach space of measurable functions $L^1(\mathbb{R}_+,\mathbb{B}) = \Big\{f:\mathbb{R}_+\to\mathbb{B} : ||f||_* := \int_{\mathbb{R}_+}||f(t)||\mathrm{d}t < \infty\Big\}$

$$\text{Then} \qquad \mathcal{P}_s: \quad (\mathcal{P}_s f)(t) := U_{t,t+s}(f(t+s)), \qquad (1.2)$$

$$\text{and} \qquad \mathcal{Q}_s: \quad (\mathcal{Q}_s f)(t) := \mathrm{e}^{s\cdot C(t)}(f(t)),\ \forall t,s\geq 0, \qquad (1.3)$$

define continuous one-parameter *semi*-groups of '*space-time*' operators on $\widetilde{\mathbb{B}} := (L^1(\mathbb{R}_+,\mathbb{B}),||\cdot||_*)$, where $(\mathcal{P}_s)_{s\geq 0}$ are contractions and $|||\mathcal{Q}_s||| \leq \mathrm{e}^{s\cdot\beta}$, $s\geq 0$, $|||\cdot|||$ denoting the operator norm on $\widetilde{\mathbb{B}}$. See e.g., [23], II.7, [11], 8.6, 8.7 for the space-time semigroup (1.2), with $\widetilde{\mathbb{B}} := C_0(\mathbb{R}_+,\mathbb{B})$. Here, to ensure $\mathcal{Q}_s\widetilde{\mathbb{B}} \subseteq \widetilde{\mathbb{B}}$ in (1.3), we had to use $\widetilde{\mathbb{B}} := L^1(\mathbb{R}_+,\mathbb{B})$.

Let $\mathbb{T}$ and $\mathbb{S}$ denote the generators of $(\mathcal{P}_s)_{s\geq 0}$ and $(\mathcal{Q}_s)_{s\geq 0}$ respectively. In particular, $\mathbb{S}:\ (\mathbb{S}f)(t) := C(t)(f(t)),\ t\geq 0$, is a bounded operator. Let $(\mathcal{R}_s)_{s\geq 0}$ denote the semigroup generated by $\mathbb{T}+\mathbb{S}$. (The addition of generators is well defined since $\mathbb{S}$ is bounded.)

According to T. Kato [18], IX, §2, Theorem 2.1, (2.4), (2.5), (resp. [14], (13.2.4)–(13.2.6), [25], [23], II.3, [5], I, 6.4), $(\mathcal{R}_s)_{s\geq 0}$ is representable by a norm-convergent *perturbation series* in $\mathcal{B}(\widetilde{\mathbb{B}})$:

$\mathcal{R}_s = \sum\limits_{k\geq 0}\mathfrak{V}^{(k)}_s$ where $\mathfrak{V}^{(0)}_s = \mathcal{P}_s$ and $\mathfrak{V}^{(k+1)}_s = \int_0^s \mathcal{P}_u\,\mathbb{S}\mathfrak{V}^{(k)}_{s-u}\mathrm{d}u$.

(Equivalently, $\mathfrak{V}^{(k+1)}_s = \int_0^s \mathcal{P}_{s-u}\,\mathbb{S}\mathfrak{V}^{(k)}_u\mathrm{d}u$, cf. e.g., [18], [14].)

Let $f\in\widetilde{\mathbb{B}},\ k\geq 0,\ t,s\geq 0,\ 0\leq u\leq s$.

Claim: $\forall t,s\geq 0, k\in\mathbb{Z}_+$ there exist operators $V^{(k)}_{t,t+s}\in\mathcal{B}(\mathbb{B})$ such that

$$\left(\mathfrak{V}^{(k)}_s f\right)(t) = V^{(k)}_{t,t+s}(f(t+s)) \quad \lambda^1-\text{a.e.} \qquad (1.4)$$

$[\![$ $k=0:$ $\quad \left(\mathfrak{V}_s^{(0)}f\right)(t)=\left(\mathcal{P}_s f\right)(t)=U_{t,t+s}\left(f(t+s)\right)$, hence the assertion with $V_{t,t+s}^{(0)}=U_{t,t+s}$.

$k+1>0:$ $\quad$ Assume that (1.4) is proved for $k'\leq k$. Then

$$\begin{aligned}\left(\mathfrak{V}_r^{(k+1)}f\right)(w) &= \int_0^r\left(\mathfrak{V}_u^{(0)}\mathbb{S}\mathfrak{V}_{r-u}^{(k)}f\right)(w)\mathrm{d}u\\ &= \int_0^r U_{w,w+u}\left(h_k(w+u)\right)\mathrm{d}u =: (*),\end{aligned}$$

where $h_k(w'):=C(w')(g_k(w'))$, $g_k(w'):=V_{w',w'+r-u}\left(f(w'+r-u)\right)$.

For $w':=w+u$ we obtain therefore

$(*)=\int_0^r U_{w,w+u}C(w+u)V_{w+u,w+r}\left(f(w+r)\right)\mathrm{d}u.$ Inserting $r=s, w=t$ this yields

$$\begin{gathered}\left(\mathfrak{V}_s^{(k+1)}f\right)(t)=\\ \int_0^s U_{t,t+u}C(t+u)V_{t+u,t+s}^{(k)}\left(f(t+s)\right)\mathrm{d}u =: V_{t,t+s}^{(k+1)}\left(f(t+s)\right)\end{gathered}$$

Put $f=\varphi\otimes x$, $x\in\mathbb{B},\varphi\in L^1(\mathbb{R}_+)$, i.e., $f:t\mapsto\varphi(t)x$, where $0\leq\varphi\leq 1$, and $\varphi\equiv 1$ on $[a,b]$. Then for $s,t,s+t\in[a,b]$ we obtain:

$V_{t,t+s}^{(k+1)}\left((\varphi\otimes x)(s+t)\right)=V_{t,t+s}^{(k+1)}(x)=\int_0^s U_{t,t+u}C(t+u)V_{t+u,t+s}^{(k)}(x)\mathrm{d}u$, as asserted. $]\!]$

Note that (1.4) holds true for λ^1-a almost all t. But considering the particular $f:=\varphi\otimes x$ as above, continuity of $(t,t+s)\mapsto U_{t,t+s}(x)$ $(\forall x)$ yields that $(t,t+s)\mapsto V_{t,t+s}^{(k)}(x)$ is continuous $(\forall x$ and $\forall k.)$ Hence for $f=\psi\otimes x$, $\psi\in L^1\cap C_0(\mathbb{G})$, (1.4) is valid for all $t\geq 0$.

Note that $V_{t,t+u}^{(0)}=U_{t,t+u}$, $V_{t',t'+s'}^{(1)}=\int_0^{s'}U_{t',t'+u_1}C(t'+u_1)U_{t'+u_1,t'+s'}\,\mathrm{d}u_1$, hence, inserting $t'=t+u, s'=s-u$

$$\begin{gathered}V_{t,t+s}^{(2)}=\\ \int_0^s\int_0^{s-u}U_{t,t+u}C(t+u)U_{t+u,t+u+u_1}C(t+u+u_1)U_{t+u+u_1,t+s}\,\mathrm{d}u_1\mathrm{d}u\end{gathered}$$

whence by induction

$$\begin{aligned}V_{t,t+s}^{(k+1)} = \int_0^s\int_0^{w_0}\cdots\int_0^{w_k} & U_{t,t+v_0}C(t+v_0)\cdots\\ & \cdots U_{t+v_k}C(t+v_{k+1})U_{t+v_{k+1},t+s}\mathrm{d}u_{k+1}\cdots\mathrm{d}u_1\mathrm{d}u\end{aligned}$$

where $v_0:=u, v_i:=u+\sum_1^i u_j, w_i:=s-v_i$.

Whence immediately $||V_{t,t+s}^{(k)}||\leq\frac{s^k\beta^k}{k!}$ follows, hence $||V_{t,t+s}||\leq\mathrm{e}^{\beta s}$.

Finally, the relations $\mathcal{R}_s\,(\varphi\otimes x)\,(t) = \left(\sum_k V^{(k)}_{t,t+s}(x)\right)\cdot\varphi(t+s) =: V_{t,t+s}(x)\cdot\varphi(t+s)$ and furthermore $\mathcal{R}_s\mathcal{R}_{s'} = \mathcal{R}_{s+s'}$ yield the hemigroup property $V_{t,t+s+s'}$
$= V_{t,t+s}V_{t+s,t+s+s'}$. (Here, φ, s, s', t are suitably chosen as above.)

b) **Claim:** Let $x\in\mathbb{D}$ then
$\frac{\mathrm{d}^+}{\mathrm{d}s}V_{t,t+s}(x)|_{s=0} = \sum_k \frac{\mathrm{d}^+}{\mathrm{d}s}V^{(k)}_{t,t+s}(x)|_{s=0} = A(t)(x) + C(t)(x)$

$\llbracket$ $k=0:$ By assumption, $\frac{\mathrm{d}^+}{\mathrm{d}s}V^{(0)}_{t,t+s}(x)|_{s=0} = \frac{\mathrm{d}^+}{\mathrm{d}s}U_{t,t+s}(x)|_{s=0} = A(t)(x)$ for $x\in D(A(t))$.

Furthermore, for $f\in D(\mathbb{T})$ we have $\frac{\mathrm{d}^+}{\mathrm{d}s}\mathcal{R}_s f|_{s=0} = \mathbb{T}f + \mathbb{S}f$.

If $x\in\mathbb{D}$ and $\varphi\in C^1\cap L^1(\mathbb{R}_+)$ then $f := \varphi\otimes x\in D(\mathbb{T})$, and $(\mathbb{T}f)\,(t) = \frac{\mathrm{d}^+}{\mathrm{d}s}\left(U_{t,t+s}(x)\cdot\varphi(t+s)\right)|_{s=0} = A(t)(x)\cdot\varphi(t) + x\cdot\varphi'(t)$. On the other hand, $\mathbb{S}(\varphi\otimes x)(t) = C(t)(x)\varphi(t)$. Moreover, $\frac{\mathrm{d}^+}{\mathrm{d}s}\mathrm{e}^{s\mathbb{S}}|_{s=0} = \mathbb{S}$ is bounded, hence we obtain for λ^1-almost all t

$$\begin{aligned}\frac{\mathrm{d}^+}{\mathrm{d}s}\mathcal{R}_s\,|_{s=0}(\varphi\otimes x)(t) &= \frac{\mathrm{d}^+}{\mathrm{d}s}\left(V_{t,t+s}(x)\varphi(t+s)\right)|_{s=0} = \\ &= \frac{\mathrm{d}^+}{\mathrm{d}s}\left((U_{t,t+s}(x)\cdot\varphi(t+s)\right)|_{s=0} + C(t)(x)\cdot\varphi(t) \\ &= x\cdot\varphi'(t) + \left(A(t)+C(t)\right)(x)\cdot\varphi(t)\end{aligned}$$

Whence the assertion follows if we choose φ and $t, t+s$ suitable as before. $\rrbracket$

c) The special case $C(t) = c(t)\,(S(t)-I)$:

Put $\mathbb{S} =: \widetilde{\mathbb{S}} - \beta I$, i.e. define $\widetilde{C}(t) := c(t)S(t) + (\beta - c(t))\cdot I$ and $\widetilde{\mathbb{S}} : t\mapsto \widetilde{C}(t)\,(f(t))$. Denote by $(\mathcal{R})_{s\geq 0}$ the semigroup generated by $\mathbb{T}+\widetilde{\mathbb{S}}$ and represent $\widetilde{\mathcal{R}}_s$ by a perturbation series. In view of $\mathcal{R}_s = \widetilde{\mathcal{R}}_s\cdot\mathrm{e}^{-s\cdot\beta}$, the assertion follows. □

Remarks 1.3. *Of course it is possible to obtain perturbation series representations under weaker conditions. For operator* semi*groups see e.g.,* [14], [22], [5] *or* [30] *and the references therein. Therefore, in particular, the assumptions guaranteeing that the space-time semigroups* $(\mathcal{P}_s)$ *and* $(\mathcal{Q}_s)$ *(cf. (1.2), (1.3) consists of contractions and that the generator of* $(\mathcal{Q}_s)$ *is bounded could be weakened. But in view of the applications we have in mind, both conditions appear natural. In particular, we need in the sequel that all operators* $C(t)$ *are bounded.*

2. CONTINUOUS HEMIGROUPS OF PROBABILITIES AND PERTURBATION SERIES

In the following let $\mathbb{G}$ denote a locally compact topological group. $\mathbb{G}$ is assumed to be second countable. By $\mathcal{M}^1(\mathbb{G})$ we denote the convolution semigroup of probabilities, $\star$ denotes convolution. We use the abbreviation $\langle \nu, f\rangle := \int_{\mathbb{G}} f \mathrm{d}\nu$.

In the sequel we apply the results of Section 1 to operators defined by convolution hemigroups on a locally compact group. (Cf. Definition 2.1 below). There, $\mathbb{B} := C_0(\mathbb{G})$ and $\mu \in \mathcal{M}^b(\mathbb{G})$ is identified with the convolution operator $R_\mu : R_\mu f(x) := \int_{\mathbb{G}} f(xy)\mathrm{d}\mu(y)$, $f \in C_0(\mathbb{G})$.

Definition 2.1. ***a)*** *A continuous convolution semigroup is a one-parameter family of probabilities* $(\mu_s)_{s\geq 0}$ *depending continuously on* s, *and satisfying* $\mu_{s+t} = \mu_s \star \mu_t$ *for all* $s,\ t \geq 0$. *Throughout we assume* $\mu_0 = \varepsilon_0$.
b) *(Cf.* [29], [10], [11].*) A* ***convolution hemigroup*** *is a two-parameter family of probabilities* $(\mu_{t,t+s})_{0\leq t\leq t+s\leq T}$, *depending continuously on the time parameters* $(t, t+s)$ *and fulfilling* $\mu_{t,t+s}\star\mu_{t+s,t+s+s'} = \mu_{t,t+s+s'}$ *where* $0 \leq t \leq t + s \leq t + s + s' \leq T$, *for some* $0 < T \leq \infty$.

If $(\mu_{t,t+s})_{0\leq t\leq t+s\leq T}$ is a convolution hemigroup of probabilities then the convolution operators $\left(U_{t,t+s} := R_{\mu_{t,t+s}}\right)_{0\leq t\leq t+s\leq T}$ form a continuous hemigroup of contractions on the Banach space $\mathbb{B} := C_0(\mathbb{G})$.

We will frequently make use of the following well-known observation:

Lemma 2.2. *Let* $(\mu_{t,t+s})_{0\leq t\leq t+s}$ *be a separately continuous hemigroup, i.e.,* $t \mapsto \mu_{s,t}$ *and* $s \mapsto \mu_{s,t}$ *are continuous, and* $\mu_{t,t} = \varepsilon_e$ *for all* t. *Then* $\forall T < \infty$, *for all sequences* $0 \leq t_n \leq t_n+s_n \leq T$ *with* $s_n \to 0$ *we obtain:* $\mu_{t_n,t_n+s_n} \to \varepsilon_e$ *stochastically.*

Consequently, for all neighbourhoods U *of* e *and all* $s_n \to 0$ *we obtain:* $\sup\limits_{0\leq t\leq T} \mu_{t,t+s_n}(\complement U) \to 0$.

Proof. For all subsequences $(n') \subseteq \mathbb{N}$ there exists a converging subsequence $(n'') \subseteq (n')$, i.e., $t_n \xrightarrow{(n'')} t_0 \in [0, T]$. Hence $\forall r > t_0$ we have $r \geq t_n + s_n$ for sufficiently large $n \geq n(r)$ and by continuity, $\mu_{t_n,t_n+s_n} \star \mu_{t_n+s_n,r} \to \mu_{t_0,r}$ along (n''), and also $\mu_{t_n+s_n,r} \to \mu_{t_0,r}$. Whence by the shift-compactness theorem ([24], III, Theorem 2.1, 2.2, [10], Theorem 1.21) we obtain that $\{\mu_{t_n,t_n+s_n}\}$ is relatively compact and all accumulation points ν satisfy $\nu \star \mu_{t_0,r} = \mu_{t_0,r}$. Hence, considering $r = r_n \searrow t_0$, it follows $\nu \star \varepsilon_e = \varepsilon_e$, whence $\nu = \varepsilon_e$.

Hence we have shown: For all subsequences $(n') \subseteq \mathbb{N}$ there exists a subsequence $(n'') \subseteq (n')$ such that $\mu_{t_n, t_n+s_n} \to \varepsilon_e$ along (n''). Whence the assertion follows. $\square$

Corollary 2.3. *For a hemigroup* $(\mu_{t,t+s})$ *as above we obtain: For all functions* $\varphi \in C^b(\mathbb{G})_+$ *for all* $\varepsilon > 0$, $0 < T < \infty$ *there exists a* $\delta = \delta(\varepsilon, T) > 0$ *such that for* $0 \leq t \leq t+s \leq T$, $s \leq \delta$ *it follows* $\langle \mu_{t,t+s}, \varphi \rangle \geq \varphi(e) - \varepsilon$.

Let $(\mu_t)_{t \geq 0}$ be a continuous convolution semigroup with corresponding C_0-contraction semigroup (R_{μ_t}) acting on $C_0(\mathbb{G})$. The infinitesimal generator is defined as $N := \frac{\mathrm{d}^+}{\mathrm{d}t} R_{\mu_t} |_{t=0}$. Then $D(N) \supseteq \mathcal{D}(\mathbb{G})$, the Schwartz-Bruhat space, and moreover, $\mathcal{D}(\mathbb{G})$ is a core for N. The *generating functional* is defined as $\langle A, f \rangle := Nf(e) = \frac{\mathrm{d}^+}{\mathrm{d}t} \langle \mu_t, f \rangle |_{t=0}$ for $f \in \mathcal{D}(\mathbb{G})$. In fact, A is canonically extended to $\mathcal{E}(\mathbb{G}) := \{ f \in C^b(\mathbb{G}) : f \cdot \varphi \in \mathcal{D}(\mathbb{G}) \ \forall \varphi \in \mathcal{D}(\mathbb{G}) \}$. (For details see e.g., [10], IV, 4.1–4.5. Note that for Lie groups we have $\mathcal{D}(\mathbb{G}) = C_c^\infty(\mathbb{G})$ and $\mathcal{E}(\mathbb{G}) = C_b^\infty(\mathbb{G})$). As a consequence of E. Siebert's characterization of generating functionals ([26], Satz 5, [10], 4.4.18, 4.5.8) we obtain for Lipschitz-continuous hemigroups $(\mu_{t,t+s})$ that $\frac{\mathrm{d}^+}{\mathrm{d}s} \langle \mu_{t,t+s}, f \rangle |_{s=0} =: \langle A(t), f \rangle$ exists λ^1- a.e. and defines a family of generating functionals $(A(t))_{0 \leq t \leq T}$. (For details see e.g., [29], Theorem 4.3, Corollary 4.5., [11], [12].)

$(\mu_{t,t+s})$ is a priori defined for $0 \leq t \leq t+s \leq T$ (for some $T \leq \infty$). If the hemigroup is (a.e.) differentiable with generating functionals $A(t) = \frac{\partial^+}{\partial s} \mu_{t,t+s}|_{s=0}$ and if $T < \infty$ we continue tacitely the hemigroup beyond time T defining $A(T+t) := A(t)$ resp. $\mu_{T+t,T+t+s} = \mu_{t,t+s}, 0 \leq t \leq T$, etc.

Next we apply the results in Section 1 to convolution hemigroups. Tacitly we identify measures with convolution operators on $\mathbb{B} := C_0(\mathbb{G})$ and we identify the generating functionals of continuous convolution semigroups with generators of the corresponding C_0-contraction semigroups.

We note the following corollaries to Theorem 1.2:

Corollary 2.4. *Let* $(\mu_{t,t+s})_{0 \leq t \leq t+s}$ *be a Lipschitz-continuous hemigroup in* $\mathcal{M}^1(\mathbb{G})$ *with a family of generating functionals* $A(t) = \frac{\partial^+}{\partial s} \mu_{t,t+s}|_{s=0}$, *for* λ^1*-almost all* t. *(For details the reader is referred e.g., to* [28], [29], [11].*) Let, for* $t \geq 0$, $\gamma(t) := c(t) \cdot (\rho(t) - \varepsilon_e)$ *be Poisson generators, where* $\rho(t) \in \mathcal{M}^1(\mathbb{G})$ *and* $0 \leq c(t) \leq \beta < \infty$. *Furthermore,* $t \mapsto c(t)$ *and* $t \mapsto \rho(t) \in \mathcal{M}^1(\mathbb{G})$ *are assumed to be measurable.*

Then there exists an a.e. differentiable hemigroup $(\nu_{t,t+s})$ *with generating functionals* $\frac{\partial^+}{\partial s} \nu_{t,t+s}|_{s=0} = A(t) + \gamma(t)$ *for* λ^1 *a.a.* $t \geq 0$.

$\nu_{t,t+s}$ *admits a representation by perturbation series :*

$$\nu_{t,t+s} = \mathrm{e}^{-\beta\cdot s}\sum_{k\geq 0}\nu^{(k)}_{t,t+s}$$

where $\nu^{(0)}_{t,t+s} = \mu_{t,t+s}$, $\nu^{(k+1)}_{t,t+s} = \int_0^s \mu_{t,t+u}\star\sigma(t+u)\star\nu^{(k)}_{t+u,t+s}\mathrm{d}u$, *and* $\sigma(r) := c(r)\rho(r) + (\beta - c(r))\cdot\varepsilon_e \in \mathcal{M}^b_+(\mathbb{G})$.

Furthermore, $\nu^{(k)}_{t,t+s} \in \mathcal{M}^b_+(\mathbb{G})$ *with* $||\nu^{(k)}_{t,t+s}|| \leq \frac{\beta^k\cdot s^k}{k!}$ *for* $k \geq 0$.

Proof. Immediate consequence of Theorem 1.2 c), since $||\sigma(r)|| = \beta$ and $||\mu_{t,t+u}\star\sigma(t+u)\star\nu^{(k)}_{t+u,t+s}|| = \beta\cdot||\nu^{(k)}_{t+u,t+s}||$, for all $0 \leq t \leq t+u \leq t+s$, $k \in \mathbb{Z}_+$. □

In particular we are interested in the following *special case*:

Corollary 2.5. *Let* $(\nu_{t,t+s})_{0\leq t\leq t+s}$ *be a Lipschitz-continuous hemigroup in* $\mathcal{M}^1(\mathbb{G})$ *with generating functionals* $A(t) = \frac{\partial^+}{\partial s}|_{s=0}\nu_{t,t+s}$, *for* λ^1-*almost all* t. *Let* U *be an open neighbourhood of* e *in* $\mathbb{G}$ *such that the Lévy measures satisfy*

$$\eta_{A(t)}(\complement U) =: c(t) \leq \beta < \infty \quad \textit{for all} \quad t \tag{2.1}$$

and $t \mapsto A(t)$, *hence* $t \mapsto c(t)$ *are measurable. Put* $\gamma(t) := c(t)\,(\rho(t) - \varepsilon_e)$ *with* $\rho(t) := \frac{1}{c(t)}\eta_{A(t)}|_{\complement U} \in \mathcal{M}^1(\mathbb{G})$ *and put* $\overline{A}(t) := A(t) - \gamma(t)$. *Let finally* $(\mu_{t,t+s})$ *be the hemigroup generated by* $\big(\overline{A}(t)\big)$, $t \geq 0$.

Then $(\nu_{t,t+s})$ *admits a series representation*

$$\nu_{t,t+s} = \mathrm{e}^{-\beta s}\sum_{k\geq 0}\nu^{(k)}_{t,t+s}$$

with summands $\nu^{(k)}_{t,t+s}$ *sharing the properties described in Corollary 2.4*

$\llbracket$ Put $\gamma(t) := \eta_{A(t)}|_{\complement U} - \eta_{A(t)}(\complement U)\cdot\varepsilon_e = c(t)\,(\rho(t) - \varepsilon_e)$, hence $\sigma(t) = \eta_{A(t)}|_{\complement U} + \big(\beta - \eta_{A(t)}(\complement U)\big)\cdot\varepsilon_e$ and apply Corollary 2.4. $\rrbracket$

3. Sub-multiplicative and sub-additive functions

We collect some properties of sub-multiplicative and sub-additive functions. At first we note the nearly obvious

Lemma 3.1. *Let* $f : \mathbb{G} \to \mathbb{R}_+$ *be sub-multiplicative and* $g : \mathbb{G} \to \mathbb{R}_+$ *sub-additive. Then*

a) *If* $f \neq 0$ *then* $f(e) \geq 1$. *If* $f \neq 0$ *and symmetric, i.e.,* $f(x^{-1}) = f(x)\ \forall x$ *then* $f \geq 1$. *(In fact, as immediately seen,* $f \geq \sqrt{f(e)}$*)*

b) $k := f + 1$ *and* $h := g + 1$ *are sub-multiplicative and* ≥ 1.

c) $h := \mathrm{e}^g$ *is sub-multiplicative and* ≥ 1.

d) *If* $f \geq 1$ *then* $h := \log f$ *is sub-additive and* ≥ 0. *Hence according to b),* $\log(g+1)+1$ *is sub-multiplicative and* ≥ 1.
e) *If* $f \geq 1$ *then* $\widetilde{f} : x \mapsto f(x^{-1})$ *is sub-multiplicative and* ≥ 1. *Furthermore,* $h := \max\left(f, \widetilde{f}\right)$ *is sub-multiplicative,* ≥ 1 *and symmetric.*
f) *Let* $\mathbb{G}$ *be second countable and let* f *be measurable with* $f(e) \geq 1$. *Then the function* $F : x \mapsto \sup_{y\in\mathbb{G}} (f(xy)/f(y))$ *is sub-multiplicative, measurable with* $F(e) = 1$ *and satisfying* $F \leq f \leq f(e) \cdot F$.

In view of Lemma 3.1 there is no serious loss of generality if we restrict in the following frequently to a particular class of sub-multiplicative functions f:

Definition 3.2. *A sub-multiplicative function* f *is called* admissible *if* f *is continuous, symmetric,* $f \geq 1$ *with* $f(e) = 1$.
Analogously, a sub-additive function g *is called* admissible *if* g *is continuous, symmetric,* $g \geq 0$ *with* $g(e) = 0$.

Lemma 3.3. $g(xy) \geq |g(x) - g(y)|$ *for all* $x, y \in \mathbb{G}$ *if* g *is sub-additive, symmetric and* ≥ 0. *Hence, if* g *is continuous at* e *with* $g(e) = 0$ *then* g *is (left- and right) uniformly continuous:*
$\max\left(|g(xy) - g(x)|\ ,\ |g(yx) - g(x)|\right) \leq g(y)\ \forall x, y \in \mathbb{G}$.

$\Big[\!\Big[$ $g(x) = g\left((xy)y^{-1}\right) \leq g(xy) + g(y)$ and on the other hand, we have $g(y) = g\left(x^{-1}(xy)\right) \leq g(x) + g(xy)$. Whence the assertion. $\Big]\!\Big]$

Proposition 3.4. *Let* $f : \mathbb{G} \to [1, \infty)$ *be sub-multiplicative and symmetric. Then we have:*

$$f(xy) \geq \frac{f(x)}{f(y)} \cdot 1_{\{f(x)\geq f(y)\}} + \frac{f(y)}{f(x)} \cdot 1_{\{f(y)>f(x)\}}$$

Whence in particular, $f(xy) \geq \max\left\{\frac{f(x)}{f(y)}, \frac{f(y)}{f(x)}, 1\right\}$

$\Big[\!\Big[$ Applying Lemma 3.3 to $g := \log f$ yields:
$f(xy) = \mathrm{e}^{g(xy)} \geq \mathrm{e}^{|g(x)-g(y)|} = \frac{f(x)}{f(y)} \cdot 1_{\{f(x)\geq f(y)\}} + \frac{f(y)}{f(x)} \cdot 1_{\{f(y)>f(x)\}}$ $\Big]\!\Big]$

Proposition 3.5. *Let* $f : \mathbb{G} \to [1, \infty)$ *be measurable, symmetric and sub-multiplicative. Let* $\mu, \nu, \lambda \in \mathcal{M}^b_+(\mathbb{G})$. *Then we have:*
a) $\langle \mu \star \nu, f\rangle \leq \langle \mu, f\rangle \cdot \langle \nu, f\rangle$
b) $\langle \mu \star \nu, f\rangle \geq \max\left\{\langle \mu, f\rangle \cdot \langle \nu, 1/f\rangle\ \ ,\ \langle \mu, 1/f\rangle \cdot \langle \nu, f\rangle\right\}$
c) *Hence* $\langle \mu \star \nu \star \lambda, f\rangle \geq$
$\max\left\{\langle \mu, f\rangle \cdot \langle \nu, \frac{1}{f}\rangle \cdot \langle \lambda, \frac{1}{f}\rangle\ ,\ \langle \mu, \frac{1}{f}\rangle \cdot \langle \nu, f\rangle \cdot \langle \lambda, \frac{1}{f}\rangle\ ,\ \langle \mu, \frac{1}{f}\rangle \cdot \langle \nu, \frac{1}{f}\rangle \cdot \langle \lambda, f\rangle\right\}$

Proof. a) is obvious.

$$
\begin{aligned}
b) \qquad \langle \mu \star \nu, f \rangle &= \int \int f(xy) d\mu(x) \mathrm{d}\nu(y) \\
&\overset{Prop. 3.4}{\geq} \int \int \frac{f(x)}{f(y)} \cdot 1_{\{f(x) \geq f(y)\}} + \frac{f(y)}{f(x)} \cdot 1_{\{f(y) > f(x)\}} \mathrm{d}\nu(y) \mathrm{d}\mu(x) \\
&= \int f(x) \int \frac{1}{f(y)} \left(1_{\{f(x) \geq f(y)\}} + \frac{f(y)^2}{f(x)^2} \cdot 1_{\{f(y) > f(x)\}} \right) \mathrm{d}\nu(y) \mathrm{d}\mu(x) \\
&\geq \int f(x) \int \frac{1}{f(y)} \left(1_{\{f(x) \geq f(y)\}} + 1_{\{f(y) > f(x)\}} \right) \mathrm{d}\nu(y) \mathrm{d}\mu(x) \\
&= \langle \mu, f \rangle \cdot \langle \nu, 1/f \rangle
\end{aligned}
$$

The other assertions are now obvious. □

Proposition 3.6. *Let f be sub-multiplicative admissible (cf. Definition 3.2), assume $\mu_n \to \mu$ weakly in $M^b_+(\mathbb{G})$. Then $\langle \mu, f \rangle \leq \liminf \langle \mu_n, f \rangle$*

⟦ For all $N > 0$ we have $\langle \mu_n, f \wedge N \rangle \to \langle \mu, f \wedge N \rangle$ by assumption, hence $\langle \mu, f \rangle = \sup_N \langle \mu, f \wedge N \rangle = \sup_N \lim_n \langle \mu_n, f \wedge N \rangle \leq \liminf_n \langle \mu_n, f \rangle$ ⟧

Proposition 3.7. *Let $f : \mathbb{G} \to [1, \infty)$ be sub-multiplicative and admissible. Let $(\mu_{t,t+s})_{0 \leq t \leq t+s}$ be a continuous hemigroup with $\langle \mu_{t_0,t_0+s_0}, f \rangle < \infty$. Then $\sup\limits_{t_0 \leq t \leq t+s \leq t_0+s_0} \langle \mu_{t,t+s}, f \rangle < \infty$.*

Proof. Let $\alpha \in (0, 1)$. Then there exist a $\delta = \delta(\alpha) > 0$ such that for $0 < u - v < \delta$ we have $\langle \mu_{u,v}, 1/f \rangle > \alpha$ (cf. Lemma 2.2, Corollary 2.3 applied to $\varphi = 1/f$). Furthermore, according to Lemma 3.5 we have $\langle \mu_{t_0,t_0+s_0}, f \rangle \geq \langle \mu_{t_0,t_0+v}, 1/f \rangle \langle \mu_{t_0+v,t_0+u}, f \rangle \langle \mu_{t_0+u,t_0+s_0}, 1/f \rangle$.

Consequently, choose t_1, s_1 such that $t_0 \leq t_1 \leq t_1 + s_1 \leq t_0 + s_0 < \delta$, $t_1 - t_0 < \delta$ and $t_0 + s_0 - t_1 - s_1 < \delta$, then $\langle \mu_{t_1,t_0+s_0}, f \rangle \leq \langle \mu_{t_0,t_0+s_0}, f \rangle \cdot \alpha^{-1}$, $\langle \mu_{t_0,t_1+s_1}, f \rangle \leq \langle \mu_{t_0,t_0+s_0}, f \rangle \cdot \alpha^{-1}$, and $\langle \mu_{t_1,t_1+s_1}, f \rangle \leq \langle \mu_{t_0,t_0+s_0}, f \rangle \cdot \alpha^{-2}$.

Let $[t_*, t_* + s_*] \subseteq [t_0, t_0 + s_0]$ be a sub-interval of length $s_* < \delta$. Then there exist $t_0 < \cdots < t_i < t_{i+1} < \ldots t_{N+1} := t_0 + s_0$ such that $t_{i+1} - t_i < \delta \ \forall i$ and $t_* = t_{i_0}, t_* + s_* = t_{i_0+1}$ for some i_0. Therefore, repeating the above consideration $N-$times, we obtain $\langle \mu_{t_*,t_*+s_*}, f \rangle \leq \langle \mu_{t_0,t_0+s_0}, f \rangle \cdot \alpha^{-2N}$.

Hence for any sub-interval $[t, t + s] \subseteq [t_0, t_0 + s_0]$, decomposing $[t, t + s]$ in at most N sub-intervals of lengths $\leq \delta$ we obtain finally $\langle \mu_{t,t+s}, f \rangle \leq \left(\langle \mu_{t_0,t_0+s_0}, f \rangle \cdot \alpha^{-2N} \right)^N$. (Note that $N \approx [s_0/\delta] + 1$ can be chosen independently from the particular decomposition.) □

4. MOMENTS OF LIPSCHITZ-CONTINUOUS HEMIGROUPS AND THEIR LÉVY-MEASURES

The following key-result is proved in [28], Theorem 4:

Proposition 4.1. *Let* $(\mu_{t,t+s})$, $t, s \geq 0$, *be a Lipschitz-continuous hemigroup with generating functionals* $(A(t))$, *resp.* $B(s,t) := \int_s^t A(\tau)\mathrm{d}\tau$ *and Lévy measures* $\eta_{A(\tau)}$ *and* $\eta_{B(s,t)} = \int_s^t \eta_{A(\tau)}\mathrm{d}\tau$ *respectively.*

Assume that there exists a neighbourhood U *of* e *such that*

$$\eta_{A(\tau)}\left(\complement U\right) = 0 \ \forall \tau, \ \textit{hence} \quad \eta_{B(s,t)}\left(\complement U\right) = 0, \ \forall\ s < t \tag{4.1}$$

Then for any continuous sub-multiplicative function $f : \mathbb{G} \to [1, \infty)$, *for all* $0 < T < \infty$ *we have:*

$$\sup_{0 \leq t \leq t+s \leq T} \langle \mu_{t,t+s}, f \rangle < \infty \tag{4.2}$$

In fact, more is shown there: Let $\alpha > 0, r \in (0, \alpha)$. Then $\exists t > 0$

$$\begin{aligned} \sup_{0 \leq s \leq t} \langle \mu_{r,r+s}, f \rangle &= \sup_{0 \leq s \leq t} \int f(X_r^{-1} X_{r+s} \mathrm{d}P \\ &\leq \int \sup_{0 \leq s \leq t} f\left(X_r^{-1} X_{r+s}\right) \mathrm{d}P \leq \beta(t). \end{aligned}$$

There $\beta(t) \searrow 1$ (with $t \searrow 0$) and $(X_r^{-1} X_{r+s})$ denote the increments of an additive process with distributions $(\mu_{r,r+s})_{r,r+s \geq 0}$.

Hence, if $f(e) = 1$, then $\sup\left(\langle \mu_{r,r+s}, f \rangle - 1\right) \to 0$. This proves in particular the assertion (4.2) if $[0, T]$ is covered by a finite number of small intervals.

Recall the notations introduced in Corollary 2.5: $c(t) = \eta_{A(t)}|_{\complement U} \leq \beta$, $c(t) \cdot \rho(t) = \eta_{A(t)}|_{\complement U}$, $\sigma(t) = c(t)\rho(t) + (\beta - c(t))\varepsilon_e$.

Lemma 4.2. *Let* $(\nu_{t,t+s})$ *be represented by a perturbation series as in Corollaries 2.4, 2.5:* $\nu_{t,t+s} = \mathrm{e}^{-\beta \cdot s} \sum_{k \geq 0} \nu_{t,t+s}^{(k)}$.

Then for sub-multiplicative admissible functions f *we have:*

a) $\langle \nu_{t,t+s}, f \rangle = \mathrm{e}^{-\beta s} \sum_{k \geq 0} \langle \nu_{t,t+s}^{(k)}, f \rangle$, $\quad \langle \nu_{t,t+s}^{(0)}, f \rangle = \langle \mu_{t,t+s}, f \rangle$ *and*

$$\begin{aligned} \langle \nu_{t,t+s}^{(k+1)}, f \rangle &\leq \int_0^s \langle \mu_{t,t+u} \star \sigma(t+u) \star \nu_{t+u,t+s}^{(k)}, f \rangle \mathrm{d}u \leq \ldots \qquad (4.3) \\ &\leq \int_0^{s_0} \cdots \int_0^{s_k} \prod_{i=0}^{k+1} \langle \mu_{t_i,t_{i+1}}, f \rangle \cdot \prod_{i=0}^{k} \langle \sigma(t_{i+1}), f \rangle \mathrm{d}u_{k+1} \cdots \mathrm{d}u_0 \end{aligned}$$

where $t_0 = t$, $t_i := t_i + u_i$, $t_{k+1} := t + s$, $s_0 := s, s_i := s - \sum_1^i u_j$.

b) $\langle \nu_{t,t+s}, f \rangle \geq \langle \mu_{t,t+s}, f \rangle \cdot \mathrm{e}^{-\beta s}$.

c) $\langle \nu_{t,t+s}, f\rangle \geq \alpha \cdot \mathrm{e}^{-\beta s} \int_0^s \langle \sigma(t+u), f\rangle \mathrm{d}u$ *for some* $\alpha = \alpha(t, t+s) \in (0, 1]$.
d) *Furthermore, we observe*
$\int_0^s \langle \sigma(t+u), f\rangle \mathrm{d}u = \int_0^s c(t+u)\langle \rho(t+u), f\rangle \mathrm{d}u + \delta(s) \cdot f(e)$ *with* $\delta(s) := \int_0^s (\beta - c(t+u))\, \mathrm{d}u \leq \beta \cdot s$.

Proof. a), and b) follow immediately by Corollaries 2.4, 2.5 and by Proposition 3.5 a).

Analogously, c) follows applying Proposition 3.5 c) to

$$\langle \nu_{t,t+s}, f\rangle \geq \mathrm{e}^{-\beta s} \int_0^s \langle \mu_{t,t+u} \star \sigma(t+u) \star \mu_{t+u,t+s}, f\rangle \mathrm{d}u$$

defining $C := \inf\limits_{0 \leq u \leq s} \langle \mu_{t,t+u}, 1/f\rangle$, $D := \inf\limits_{0 \leq u \leq s} \langle \mu_{t+u,t+s}\ , 1/f\rangle$ and $\alpha := C \cdot D$. (Recall that $f(e) = 1$ and $0 < 1/f \leq 1$.)
d) is again obvious. □

Now we have the means to formulate the main result:

Theorem 4.3. *Let* $(\nu_{t,t+s})$ *be a Lipschitz-continuous hemigroup with generating functionals* $A(t) = \frac{\partial}{\partial s}|_{s=0}\ \mu_{t,t+s}$ *and* $B(s, t) = \int_s^t A(\tau)\mathrm{d}\tau$ *respectively. Assume as in Corollary 2.5 formula (2.1)*

$$c(\tau) := \eta_{A(\tau)}\left(\complement U\right) \leq \beta, \quad 0 \leq \tau \leq T \tag{4.4}$$

for some neighbourhood U *of the unit* e. *Let as before,* $f : \mathbb{G} \to [1, \infty)$ *be sub-multiplicative admissible (cf. Definition 3.2). Then the following assertions are equivalent:*
(i) $\langle \nu_{t,t+s}, f\rangle < \infty$ *for all* $0 \leq t \leq t+s \leq T$
(ii) $\langle \nu_{0,T}, f\rangle < \infty$
(iii) $\int_0^T \langle \sigma(\tau), f\rangle \mathrm{d}\tau < \infty$ *(with the notations introduced in 2.5).*
(iv) $\langle \eta_{B(0,T)}, f 1_{\complement U}\rangle = \int_0^T \int_{\complement U} f \mathrm{d}\eta_{A(\tau)} d\tau < \infty$
(v) $\sup\limits_{0 \leq t \leq t+s \leq T} \langle \eta_{B(t,t+s)}, f 1_{\complement U}\rangle < \infty$ *for all intervals* $[t, t+s] \subseteq [0, T]$*:*

Proof. We use the notations introduced before, cf. especially Corollary 2.5.
"$(i) \Leftrightarrow (ii)$" cf. Lemma 3.7.
"$(iii) \Leftrightarrow (iv)$" Note that $\sigma(\tau) \geq 0$, $\beta T \geq \int_0^T \beta - \eta_{A(r)}(\complement U)\mathrm{d}r \geq 0$ and $\langle \eta_{B(0,T)}, f 1_{\complement U}\rangle = \int_0^T \langle \sigma(\tau), f\rangle \mathrm{d}\tau - \int_0^T \beta - \eta_{A(r)}(\complement U)\mathrm{d}r$ (cf. Lemma 4.2 d)). Whence the assertion follows.
"$(iv) \Leftrightarrow (v)$" is obvious, since the integrands are non-negative.
"$(ii) \Rightarrow (iii)$" follows by Lemma 4.2 c). (Note that $\gamma = C \cdot D > 0$).
"$(iii) \Rightarrow (ii)$" According to Lemma 4.2 a) it is sufficient to show that $\langle \nu_{0,T}, f\rangle = \mathrm{e}^{-\beta T} \sum_k \langle \nu_{0,T}^{(k)}, f\rangle < \infty$.

For $k = 0$ we have $\langle \nu^{(0)}_{0,T}, f\rangle = \langle \mu_{0,T}, f\rangle \leq \sup_{t\leq t+s\leq T}\langle \mu_{t,t+s}, f\rangle =: M_0 = M_0(T) < \infty$ (cf. Proposition 4.1.) Note that $1 \leq M_0 \leq M_0^2$ and by assumption (iii), $\int_0^T \langle \sigma(\tau), f\rangle \mathrm{d}\tau < \infty$. $t \mapsto \Gamma(t) := \int_0^t \langle \sigma(v), f\rangle \mathrm{d}v$ is increasing, bounded on $[0, T]$ and absolutely continuous w.r.t. Lebesgue measure $\lambda^1|_{[0,T]}$. Hence for all $\varepsilon > 0$ there exists a $\delta(\varepsilon) > 0$ such that $\forall\ s < \delta(\varepsilon), \forall t$ we have $\Gamma(t, t + s) := \Gamma(t + s) - \Gamma(t) < \varepsilon$. Furthermore, for $k \geq 0$, $d > 0$ we have in view of (4.3):

$$\begin{aligned}
\langle\ \nu^{(k+1)}_{t,t+s}, f\ \rangle &\overset{(4.3),\ Prop.3.5c)}{\leq} \\
&\leq \int_0^{s_0} \cdots \int_0^{s_k} \prod_{i=0}^{k+1} \langle \mu_{t_i,t_{i+1}}, f\rangle \cdot \prod_{i=0}^{k} \langle \sigma(t_{i+1}), f\rangle \mathrm{d}u_{k+1} \cdots \mathrm{d}u_0 \\
&\leq M_0^{k+2} \cdot \int_0^{s_0} \cdots \int_0^{s_k} \prod_{i=0}^{k} \langle \sigma(t_{i+1}), f\rangle \mathrm{d}u_{k+1} \cdots \mathrm{d}u_0 \\
&= M_0^{k+2} \cdot \prod_{i=0}^{k} \Gamma(0, s_k) \leq M_0 \cdot (M_0 \cdot d)^{k+1} \qquad (4.5)
\end{aligned}$$

(with the notations introduced in (4.3)), if $s < \delta(d)$, hence $s_i < \delta(d)\ \forall i$.

To prove the last estimate of (4.5) note that $\int_0^{s_k} \prod_{i=0}^{k} \langle \sigma(t_{i+1}), f\rangle \mathrm{d}u_{k+1} = \prod_{i=0}^{k-1} \langle \sigma(t_{i+1}), f\rangle \cdot \int_0^{s_k} \langle \sigma(t_k + u_{k+1}), f\rangle \mathrm{d}u_{k+1} \leq \prod_{i=0}^{k-1} \langle \sigma(t_{i+1}), f\rangle \cdot d$, etc.

Let $0 < c < 1$, choose $0 < d < c/M_0$. (Note that M_0 only depends on T.) We begin with $0 = t_0$. Put $t_{i+1} := t_i + s_i$ and choose $s_i < \delta(d)$, hence $\Gamma(t_i, t_{i+1}) < d$. Then according to (4.5) we observe $\langle \nu_{t_0,t_1}, f\rangle \leq \mathrm{e}^{-\beta s_1} \sum_k \langle \nu^{(k)}_{t_0,t_0+s_1}, f\rangle \leq \mathrm{e}^{-\beta s_1} (1 - c)^{-1} \cdot M_0$.

Now replace t_0 by $t_0 + s =: t_1,\ s < \delta(d)$ etc. After N repetitions, $N \approx T/\delta(d)$, the interval $[0, T]$ is covered and we obtain in view of Proposition 3.5: $\langle \nu_{0,T}, f\rangle \leq \prod_1^N \langle \nu_{t_i,t_{i+1}}, f\rangle \leq \mathrm{e}^{-\beta T} (1 - c)^{-N} \cdot M_0^N < \infty$. $\square$

Remarks 4.4. ***a)*** *The constant M_0 in (4.5) depends on the the length of the chosen interval: Put $M_0 = M_0(s) := \sup_{0\leq t\leq t+u\leq t+s} \langle \mu_{t,t+u}, f\rangle$ if the behaviour of $u \mapsto \nu_{t,t+u}$ is considered in the interval $0 \leq u \leq s$.*

If the hemigroup $(\nu_{t,t+s})$ is time-homogeneous, i.e. if $(\nu_s := \nu_{t,t+s})_{s\geq 0}$ (and also $(\mu_s := \mu_{t,t+s})_{s\geq 0}$) are continuous convolution semigroups, then we obtain a sharper estimate. Put with $M_0(s) = \sup_{u\leq s}\langle \mu_u, f\rangle$:

$$\langle \nu_{t,t+s}, f\rangle = \langle \nu_s, f\rangle \leq M_0(s)\mathrm{e}^{-s\beta}\mathrm{e}^{M_0(s)\langle \sigma, f\rangle} \qquad (4.6)$$

Here $A(t) \equiv A$*,* $\sigma(t) \equiv \sigma = \eta_A|_{\complement U}$*,* $\beta := \sigma(\mathbb{G}) = \eta_A(\complement U) \equiv c(t)$*.*

With different notations the upper bound (4.6) is found in [28]*, proof of Theorem 5. In fact, in the time-homogeneous case we have:*

$$\langle \nu_s, f\rangle = \langle \nu_{t,t+s}, f\rangle = \mathrm{e}^{-\beta s} \sum_k \langle \nu^{(k)}_{t,t+s}, f\rangle \quad \textit{with} \quad \langle \nu^{(0)}_{t,t+s}, f\rangle = \langle \mu_{t,t+s}, f\rangle$$

$$\langle \nu^{(k+1)}_{t,t+s}, f\rangle \le \int_0^s \langle \mu_u, f\rangle \langle \sigma, f\rangle \langle \nu^{(k)}_{t+u,t+s}, f\rangle \mathrm{d}u$$

$$\le M_0(s)\langle \sigma, f\rangle \int_0^s \langle \nu^{(k)}_{t+u,t+s}, f\rangle \mathrm{d}u \le \cdots \le \frac{M_0(s)}{(k+1)!} \left(M_0(s)\langle \sigma, f\rangle\right)^{k+1}$$

Whence (4.6) follows.

b) *E. Siebert's results in* [27], [28] *for the time-homogeneous case are proved for general continuous convolution semigroups, and in that case the restrictive condition (2.1) resp. 4.4 is trivially fulfilled (for any* $T > 0$*). (In fact, then* $\eta_A(\complement U) =: \beta < \infty$*.) It is natural to conjecture that the assertions of Theorem 4.3 hold true also without condition (2.1) resp. (4.4). But up to now no proof is available.*

c) *Throughout, in order to avoid problems with measurability and in view of* [28]*, Theorem 4, we assumed* $\mathbb{G}$ *to be second contable. In fact, this is not a serious restriction:*

At first, w.l.o.g. we may assume $\mathbb{G}$ *to be* $\sigma-$*compact, since the group generated by the supports* $\bigcup\limits_{0 \le t < t+s \le T} \mathrm{supp}(\nu_{t,t+s})$ *is* $\sigma-$*compact.*

As well known (cf. e.g., [4]*, page 101, exerc. 11) a* $\sigma-$*compact group is representable as projective limit of second countable groups* $\mathbb{G} = \lim\limits_{\leftarrow} \mathbb{G}/K$*,* $K \in \mathfrak{K}$*, a set of compact normal subgroups with* $\bigcap\limits_{K \in \mathfrak{K}} K = \{e\}$*.*

Let f *be as above, then* $W := \{f = 1\}$ *is a closed subgroup. Moreover,* $g := \log f$ *is uniformly continuous by Lemma 3.3 and* $\{g = 0\} = W$*. Hence* g *and* f *are* W*-invariant and if* g *resp.* f *is* K*-invariant for some subgroup* K*, then* $K \subseteq W$*. But* g *is* K_0-*invariant for some* $K_0 \in \mathfrak{K}$ *(cf. the above reference* [4]*), whence* $K_0 \subseteq W$*. Therefore,* f *is* K_0-*invariant, hence integrability of* f *w.r.t.* $(\nu_{t,t+s})$ *can be reduced to the case of second countable groups.*

5. Appendix

In the following we sketch briefly some applications and examples in order to show that in many interesting cases it is easier to check integrability of admissible sub-multiplicative functions w.r.t. Levy measures than w.r.t. the generated probabilities.

5.1. **Convolution semigroups: Moments of (semi-)stable laws.** Let $\mathbb{G}$ be a second countable contractible locally compact group with contracting automorphism $\tau \in \mathrm{Aut}(\mathbb{G})$. Let $\{U_k\}_{k\in\mathbb{Z}}$ be a filtration, i.e., U_k compact neighbourhoods of e with $\bigcup U_n = \mathbb{G}$, $\bigcap U_n = \{e\}$, $U_n \supseteq U_{n+1}$ and $\tau U_n = U_{n+1}$ for all $n \in \mathbb{Z}$. ([6], Lemma 3.7.3.) $L := U_0 \backslash U_1$ is a cross-section w.r.t. the action of τ. Let $(\mu_t)_{t\geq 0}$ be a (τ, c)-semistable continuous convolution semigroup, i.e., $\tau(\mu_t) = \mu_{c\cdot t}$ for all $t \geq 0$, where $0 < c < 1$. (cf., e.g., [6], §3.4).

Let $|\cdot|$ denote a sub-additive 'group-norm', i.e., a continuous symmetric sub-additive function $|\cdot| : \mathbb{G} \to \mathbb{R}_+$ such that $|x| = 0$ iff $x = e$ and $\{|x| < \varepsilon\}$ is a neighbourhood of e for $\varepsilon > 0$. ([6], 2.7.26 d).) Let, for some constants $1 < r \leq R$, $r^n|x| \leq |\tau^{-n}x| \leq R^n|x|$ for $n \in \mathbb{Z}_+$. Then, for $\gamma > 0$ we have:

$$\int |x|^\gamma \mathrm{d}\mu_t(x) < \infty, \quad t > 0 \qquad \text{iff} \qquad \int_{\{|x|\geq 1\}} |x|^\gamma \mathrm{d}\eta(x) < \infty \tag{5.1}$$

where η denotes again the Lévy measure of (μ_t). In fact, the left integral is finite iff $\int (1 + |x|^\gamma)\, \mathrm{d}\mu_t(x) < \infty$, hence iff $\int (1 + |x|)^\gamma\, \mathrm{d}\mu_t(x) < \infty$. According to Theorem 4.3 (resp. by Siebert's result) this is equivalent with $\int (1 + |x|)^\gamma\, \mathrm{d}\eta(x) < \infty$, and, as before this is the case iff $\int |x|^\gamma \mathrm{d}\eta(x) < \infty$. (Note that $x \mapsto (1 + |x|)^\gamma$ is an admissible sub-multiplicative function.)

Example 5.1. *The Lévy measure is representable as* $\eta = \sum_{k\in\mathbb{Z}} c^{-k}\tau^k(\lambda)$ *for* $\lambda = \eta|_L \in \mathcal{M}^b_+(L)$ *(cf. e.g.* [6]*, Proposition 3.4.8). Hence it follows easily that the integral in (5.1) is finite iff* $\sum_{k\geq 0} c^k \int_L |\tau^{-k}x|^\gamma \mathrm{d}\lambda(x) < \infty$*. By assumption we have* $r^k \int_L |x|^\gamma \mathrm{d}\lambda \leq \int_L |\tau^{-k}x|^\gamma \mathrm{d}\lambda \leq R^k \int_L |x|^\gamma \mathrm{d}\lambda$*. Hence we obtain:*

$$\int |x|^\gamma \mathrm{d}\mu_t(x) < \infty \quad \textit{if} \quad R^\gamma < 1/c, \ \textit{i.e.} \quad \gamma < \log(1/c)/\log R \quad \textit{and}$$

$$r^\gamma < 1/c, \ \textit{i.e.} \quad \gamma < \log(1/c)/\log r \quad \textit{if} \quad \int |x|^\gamma \mathrm{d}\mu_t(x) < \infty$$

⟦ *For vector spaces compare with e.g.,* [17], [16]*, 4.12.2–4.12.4,* [6]*, 1.7.9, for homogeneous groups* [6]*, 2.7.28–2.7.32.* ⟧

In particular, if $|\tau^{-k}x| = r^k|x|$, $k \in \mathbb{Z}_+$, then $\int |x|^\gamma d\mu_t < \infty$ iff $0 < \gamma < \log(1/c)/\log r$.

Example 5.2. The totally disconnected case
Let $\mathbb{G}$ *be totally disconnected. Then the filtration can be chosen to consist of open compact subgroups. We fix* $0 < \alpha < 1$ *and define* $|x| :=$

$\alpha^{k(x)}$ *for* $x \neq e$ *and* $|e| = 0$, *where* $k(x) := \min\{k \in \mathbb{Z} : x \in U_k\}$. *(Frequently* $\alpha := 1/p$ *where* $p := \operatorname{ord}\{U_k/U_{k+1}\}$, *the modulus of* τ.*) Then we obtain* $\int |x|^\gamma \mathrm{d}\mu_t(x) < \infty$ *iff* $\alpha^{-\gamma} < c^{-1}$, *i.e.,* $\gamma < \log c / \log \alpha$. *(In fact,* $|\cdot|_\alpha = |\cdot|$ *is a group norm, with* $|xy| \leq \max\{x, y\}$ *and* $|\tau^k x| = \alpha^k \cdot |x|,\ k \in \mathbb{Z}$. *Hence* $r = \alpha^{-1}$.*)*

Example 5.3. The case of homogeneous groups: dilation semistable laws
Let $\mathbb{G}$ *be a homogeneous group, in particular, a connected contractible Lie group with contractive automorphism* τ. *Let* $(\delta_t) \subseteq \operatorname{Aut}(\mathbb{G})$ *be a group of dilations and* $|\cdot|$ *a corresponding homogeneous norm. (Cf, e.g.,* [6], *2.7.26 d)). Assume e.g., that also* τ *is a dilation,* $\tau = \delta_d$ *for some* $0 < d < 1$. *Then we obtain as before:* $\int |x|^\gamma \mathrm{d}\mu_t(x) < \infty$ *iff* $d^{-\gamma} < c^{-1}$, *i.e.,* $\gamma < \log c / \log d$. *(Note that* $|\tau^k x| = |\delta_d^k x| = d^k \cdot |x|, \forall x \in \mathbb{G}, k \in \mathbb{Z}$ *in that case. Hence* $r = d^{-1}$.*)*

5.2. **Convolution hemigroups: Logarithmic moments of (semi-)stable hemigroups and (semi-)self-decomposability.** Let again $\mathbb{G}$ be a homogeneous group with dilations (δ_t) and corresponding subadditive homogeneous norm. Let $(\rho_t)_{t\in\mathbb{R}}$ be a contracting one-parameter group of automorphisms with additive parametrization $\rho_{t+s} = \rho_t\rho_s, \quad \rho_t(x) \overset{t\to\infty}{\to} e\ (x \in \mathbb{G})$.

Example 5.4. *Let* $(\mu_{t,t+s})_{0\leq t\leq t+s}$ *be a stable convolution hemigroup, i.e. a hemigroup satisfying* $\rho_r(\mu_{t,t+s}) = \mu_{t+r,t+s+r}$ *for all* $r, s, t \geq 0$. *(These hemigroups are the distributions of increments of an additive process, a* generalized Ornstein-Uhlenbeck *process.) It is well known that* $\lim_{t\to\infty} \mu_{0,t} =: \mu$ *exists iff logarithmic moments exist, i.e.,* $\int \log_+ |x| \mathrm{d}\mu_{0,1}(x) < \infty$, *equivalently, if* $\forall 0 \leq s, t$ $\int \log_+ |x| \mathrm{d}\mu_{t,t+s}(x) < \infty$. $[\![$ *For vector spaces see e.g.,* [16], *for groups e.g.,* [8], [6], *§2.14).* $]\!]$ *(*μ *is self-decomposable and an invariant distribution for the underlying additive process.) As before, this is equivalent with* $\int \psi(x)\mathrm{d}\mu_{t,t+s}(x) < \infty$, *where* $\psi(x) := (1 + \log(1 + |x|)) \approx \log_+(|x|)$.
The integrand ψ *is admissible sub-multiplicative, hence according to Theorem 4.3 this integral is finite iff* $\int_0^T \int_{\{|x|>1\}} \psi(x)\mathrm{d}\eta_t \mathrm{d}t < \infty$, *where* η_t *are the Lévy measures of* $\frac{\partial}{\partial s}\mu_{t,t+s}|_{s=0} =: A(t)$. *Note that the stability property implies the existence of the derivatives* $A(t) = \frac{\partial}{\partial s}|_{s=0}\ \mu_{t,t+s}$ *and furthermore,* $A(t+s) = \rho_t(A(s))$. *Hence* $A(t) = \rho_t(A(0))$ *and as* $t \mapsto \rho_t$ *is continuous, condition (2.1) resp. (4.4) is obviously fulfilled*

and we have:

$$\int_0^T \int_{\{|x|>1\}} \psi(x)\mathrm{d}\eta_t(x)\mathrm{d}t = \int_0^T \int_{\{|x|>1\}} \psi(x)\mathrm{d}\rho_t(\eta_0)(x)\mathrm{d}t < \infty$$

$$\text{iff} \qquad \int_{\{|x|>1\}} \psi(x)\mathrm{d}\eta_0(x) < \infty$$

η_0 *is the Lévy measure of the underlying background driving Lévy process.*

Hence we obtain: The additive process (X_t) *resp. its increments* $X_{t,t+s}$ *with distributions* $\mu_{t,t+s}$ *possess logarithmic moments (and hence there exists an invariant distribution* $\mu = \lim_{t\to\infty}\mu_{0,t}$*) iff the background driving Lévy process has logarithmic moments.*

Thus we obtained a new proof of a well known result: For vector spaces see e.g., [16]*, Theorem 3.6.6, for groups see e.g.,* [8, 6]*, §2.14, in particular Theorem 2.14.25.*

Example 5.5. *The proofs in* [8, 6] *rely on an embedding of* $\mathbb{G}$ *into a* space-time-*group* $\mathbb{G} \rtimes \mathbb{R}$ *and the application of E. Siebert's result to a continuous convolution semigroup on this enlarged group. This method breaks down in case of* semi-stable hemigroups *resp.* semi-self-decomposable *laws, i.e., hemigroups* $\mu_{t,t+s}$ *satisfying* $\rho(\mu_{t,t+s}) = \mu_{t+c,t+s+c}$ *for all* $t, t+s$*, some* $\rho \in \mathrm{Aut}(\mathbb{G})$ *and* $c > 0$*. Here the background driving Lévy process has to be replaced by an additive process, a* background driving additive periodic process*. For vector spaces cf.* [2]*, for groups see* [3]*. Again limits* $\mu = \lim_{t\to\infty}\mu_{0,t}$ *exists iff* $\mu_{0,c}$ *has finite logarithmic moments (equivalently – in view of Lemma 3.1– iff all* $\mu_{t,t+s}$ *share this property). Under the additional conditions that the embedding hemigroup is Lipschitz continuous and the Lévy measures* η_t *of the almost everywhere existing derivatives* $\frac{\partial}{\partial s}\mu_{t,t+s}|_{s=0} =: A(t)$ *satisfy the boundedness condition (2.1) resp. (4.4), it can be shown that the semistable hemigroup possesses logarithmic moments iff this is the case for the periodic background driving process. (For vector spaces cf. e.g.,* [2]*, 2.4, 3.2–3.4.)*

We omit the details here.

References

[1] **A. de Acosta**, *Exponential moments of vector valued random series and triangular arrays.* Ann. Probab. 8 (1980), pp. 381–389

[2] **P. Becker-Kern**, *Random integral representation of operator-semi-selfsimilar processes with independent increments.* Stochastic Process. Appl. 109 (2004), pp. 327–344

[3] **P. Becker-Kern, and W. Hazod**, *Mehler hemigroups and embedding of discrete skew convolution semigroups on simply connected nilpotent Lie groups.*

In: Infinite dimensional hatrmonic analysis IV. J. Hilgert, A. Hora, T. Kawazoe, K. Nishiyama, M. Voit ed. Proceedings Tokyo (2008). World Sci. (2008), pp. 32–46

[4] **N. Bourbaki**, *Eléments de Mathématique. Livre VI: Integration.* Chap. 7 Hermann, Paris, 1963

[5] **J.A. Goldstein**, *Semigroups of Linear Operators and Applications.* Oxford Univ. Press, Oxford, New York, 1985

[6] **W. Hazod and E. Siebert**, *Stable Probability Measures on Euclidean Spaces and on Locally Compact Groups. Structural Properties and Limit Theorems.* Mathematics and its Applications vol. 531. Kluwer A.P., 2001

[7] **W. Hazod**, *Stetige Halbgruppen von Wahrscheinlichkeitsmaßen und erzeugende Distributionen.* Lecture Notes Math. 595, Berlin-Heidelberg-New York, Springer, 1977

[8] **W. Hazod and H-P. Scheffler** *Strongly τ-decomposable and selfdecomposable laws on simply connected nilpotent Lie groups.* Mh. Math. 128 (1999), pp. 269–282

[9] **J.V. Herold and M.C. McKelvey**, *A Hille-Yosida theory for evolutions.* Israel J. Math. 36 (1980), pp. 13–40

[10] **H. Heyer**, *Probability Measures on Locally Compact Groups.* Berlin-Heidelberg-New York, Springer, 1977

[11] **H. Heyer**, Stetige Hemigruppen von Wahrscheinlichkeitsmaßen und additive Prozesse auf einer lokalkompakten Gruppe. Nieuw. Arch. Wiskunde 27 (1979), pp. 287–340

[12] **H. Heyer and G. Pap**, *Convergence of non-commutative triangular arrays of probability measures on a Lie group.* J. Theoret. Probab. 10 (1997), pp. 1003–1052

[13] **E. Heyn**, *Die Differentialgleichung $dT/dt = P(t)T$ für Operatorfunktionen.* Math. Nachrichten 24 (1962), pp. 281–380

[14] **E. Hille and R.S. Phillips**, *Functional Analysis and Semigroups.* Colloq. Publ. Amer. Math. Soc. vol. 31, Providence R.I., 1957

[15] **A. Hulanicki**, *A class of convolution semi-groups of measures on a Lie group.* In: Probability on vector spaces II. Proceedings Błazejewko (1979) pp. 82–101. Lecture Notes Math. 828 Berlin Heidelberg New York 1980

[16] **Z. Jurek and D. Mason** *Operator Limit Distributions in Probability Theory.* J. Wiley Inc., 1993

[17] **Z. Jurek and J. Smalara** *On integrability with respect to infinitely divisible measures.* Bull. Acad. Polon. Sci. 29 (1981) , pp. 179–185

[18] **T. Kato**, *Perturbation Theory for Linear Operators.* Berlin-Heidelberg-New York, Springer, 1966

[19] **T. Kato**, *Integration of the equation of evolution in a Banach space.* J. Math. Soc. Japan 5 (1953), pp. 208–234

[20] **T. Kato**, *Linear evolutions of hyperbolic type.* J. Fac. Sci. Univ. Tokyo, Sect. IA, 17 (1970), pp. 241–258

[21] **J. Kisyński**, *On semigroups generated by differential operators on Lie groups.* J. Funct. Anal. 31 (1979), pp 234–244

[22] **I. Miyadera**, *On perturbation theory of linear operators.* Tôhoku Math. Journ. 18 (1966), pp. 299–310

[23] **J. Neveu**, *Théorie des semi-groupes de Markov.* In: Univ. California Publications in Statisdtics 1 (1967), pp. 319–394

[24] **K.R. Parthasarathy**, *Probability Measures on Metric Spaces.* New York-London, Academic Press, 1967

[25] **R.S. Phillips** *Perturbation theory for semigroups of linear operators.* Trans. Amer. Math. Soc. 74 (1953), pp. 199-221

[26] **E. Siebert**, *Über die Erzeugung von Faltungshalbgruppen auf beliebigen lokalkompakten Gruppen.* Math. Z. 131 (1981), pp. 111–154

[27] **E. Siebert**, *Continuous convolution semigroups integrating a submultiplicative function.* Manuscripta Math. 37 (1982), pp. 381–391

[28] **E. Siebert**, *Jumps of stochastic processes with values in a topological group.* Prob. Math. Stat. 5 (1985), pp. 197–209

[29] **E. Siebert** *Continuous hemigroups of probability measures on a Lie group.* In: Probability measures on groups. H. Heyer ed. Proceedings Oberwolfach (1981). Lecture Notes Math. 1080 Springer (1982), pp. 362–402.

[30] **J. Voigt** *On the perturbation theory for strongly continuous semigroups.* Math. Ann. 229 (1977), 163–171

Wilfried Hazod,

Faculty of Mathematics, Technische Universität Dortmund, D-44221 Dortmund, Germany

E-mail: `Wilfried.Hazod@math.uni-dortmund.de` or `Wilfried.Hazod@tu-dortmund.de`