Blatt 12.

(H1) a) $f(x) = \ln(\sin^5(x) \cdot \cos^7(x)) = 5 \cdot \ln(\sin(x)) + 7 \cdot \ln(\cos(x))$

$$f'(x) = \frac{5 \cdot \cos x}{\sin(x)} - \frac{7 \cdot \sin(x)}{\cos(x)} = 5 \cot(x) - 7 \tan(x)$$

b) $f(x) = \arccos(x)$ $\qquad (\sin^2(x) + \cos^2(x) = 1)$

$$f'(x) = \frac{-1}{\sin(\arccos(x))} = -\frac{1}{\sqrt{1-x^2}}$$

$$(f(g(x)))' = f'(g(x)) \cdot g'(x) \, , \quad ((f \cdot g)(x))' = (f' \cdot g)(x) + (g' \cdot f)(x)$$

$$(f^{-1})'(x) = \frac{1}{f'(f^{-1}(x))}$$

(H2) a)

$$\lim_{x \to 0} \frac{1 - \cos(x/2)}{1 - \cos(x)} \overset{\text{L'Hopital}}{=} \lim_{x \to 0} \frac{1/2 \sin(x/2)}{\sin(x)} \overset{\text{L'Hopital}}{=} \lim_{x \to 0} \frac{1/4 \cos(x/2)}{\cos(x)} = \frac{1}{4}$$

b)

$$\lim_{x \to 0} \left(\frac{1}{x} - \cot(x) \right) = \lim_{x \to 0} \left(\frac{1}{x} - \frac{\cos(x)}{\sin(x)} \right)$$

$$= \lim_{x \to 0} \left(\frac{\sin(x) - x \cos(x)}{x \sin(x)} \right)$$

$$\overset{\text{L'Hopital}}{=} \lim_{x \to 0} \left(\frac{\cos(x) - \cos(x) + x \sin(x)}{\sin(x) + x \cos(x)} \right)$$

$$\overset{\text{L'Hopital}}{=} \lim_{x \to 0} \left(\frac{\sin(x) + x \cos(x)}{\cos(x) + \cos(x) - x \sin(x)} \right) = \frac{0}{2} = 0$$

c)

$$\lim_{x \to 0} \left(\frac{1}{x} - \frac{1}{\sin(x)} \right) = \lim_{x \to 0} \left(\frac{\sin(x) - x}{x \sin(x)} \right) \overset{\text{L'H.}}{=} \lim_{x \to 0} \left(\frac{\cos(x) - 1}{\sin(x) + x \cos(x)} \right)$$

$$\overset{\text{L'H.}}{=} \lim_{x \to 0} \left(\frac{-\sin(x)}{2\cos(x) - x \sin(x)} \right) = \frac{0}{2} = 0$$

d) $\lim\limits_{x \to \pi/2} (x - \pi/2) \tan(x) = \lim\limits_{x \to \pi/2} \dfrac{(x - \pi/2) \cdot \sin(x)}{\cos(x)}$

$\overset{L'H.}{=} \lim\limits_{x \to \pi/2} \dfrac{\sin(x) + (x - \pi/2)\cos(x)}{-\sin(x)} = -1$

(H3) $n \in \mathbb{N}$, $a_1 < \dots < a_n$

a) Minimalstelle von $f(x) = \left(\sum\limits_{k=1}^{n} (x-a_k)^2\right)^{1/2} \Leftrightarrow$ Minimalstelle von $h(x) = f^2(x) = \sum\limits_{k=1}^{n} (x-a_k)^2$

$h'(x) = 2 \sum\limits_{k=1}^{n} (x - a_k) \overset{!}{=} 0 \Rightarrow x^* = \frac{1}{n} \sum\limits_{k=1}^{n} a_k = \bar{a}$

$h''(x) = 2n > 0 \Rightarrow x^*$ ist Minimum von h

$\Rightarrow x^*$ ist Minimum von f ; $f(x^*) = \left(\sum\limits_{k=1}^{n} (\bar{a} - a_k)^2\right)^{1/2}$

b) Minimiere $g(x) = \sum\limits_{k=1}^{n} |x - a_k|$, g ist diffbar auf $\mathbb{R} \setminus \{a_1, \dots, a_n\}$

Für $x \in \,]a_i, a_{i+1}[$ gilt:
$$\begin{cases} g(x) = \sum\limits_{k=1}^{i} (x - a_k) + \sum\limits_{k=i+1}^{n} (a_k - x) \\ g'(x) = i + (n-i)(-1) = 2i - n \end{cases}$$

Da g stückweise diffbar und $\lim\limits_{|x| \to \infty} g(x) = \infty$, liegt ein globales Minimum bei Vorzeichenwechsel von g' vor.

1) n gerade $\Rightarrow i_0 = n/2 \in \mathbb{N}$:
$$\begin{cases} g'(x) < 0 & \text{f.a. } x < a_{i_0} \\ g'(x) = 0 & \text{f.a. } x \in [a_{i_0}, a_{i_0+1}] \\ g'(x) > 0 & \text{f.a. } x > a_{i_0+1} \end{cases}$$

Also ist jedes $x \in [a_{n/2}, a_{n/2+1}]$ ein globales Minimum.

2) n ungerade $\Rightarrow i_0 < n/2 < i_0 + 1$, $i_0 = \lfloor \frac{n}{2} \rfloor$

Da
$$\begin{cases} g'(x) < 0 & \text{f.a. } x < a_{i_0+1} \\ g'(x) > 0 & \text{f.a. } x > a_{i_0+1} \end{cases},$$

ist $x = a_{\lfloor \frac{n}{2} \rfloor + 1}$ eindeutiges globales Minimum.

(H4) $f(x) = e^{-x^2/2}$

Dann $\begin{cases} f'(x) = -\frac{2x}{2} e^{-x^2/2} = -x\, e^{-x^2/2} = -x\, f(x) \\ f(0) = e^{-0} = 1 \end{cases}$

Nun betr. $g \in C^{(1)}(\mathbb{R};\mathbb{R})$ mit

$$\begin{cases} g'(x) = -x \cdot g(x), & x \in \mathbb{R} \\ g(0) = 1 \end{cases}$$

Betr. $h(x) = g(\sqrt{x})$, $x \geq 0$. Dann

$$\begin{cases} h'(x) = \frac{1}{2\sqrt{x}}\, g'(\sqrt{x}) = -\frac{1}{2} h(x) \\ h(0) = 1 \end{cases}$$

Satz 11.21 $\Rightarrow$ $h(x) = e^{-1/2\, x}$, $x \geq 0$

Damit $g(\sqrt{x}) = e^{-1/2\, x}$, $x \geq 0$

$\Rightarrow$ $g(x) = e^{-1/2\, x^2}$, $x \geq 0$

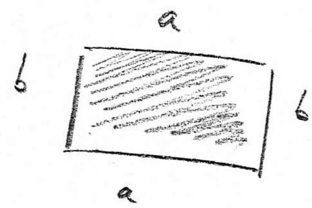
Analog: $g(x) = e^{-1/2\, x^2}$, $x \leq 0$

(Betr. $h(x) = g(\sqrt{-x})$, $x \leq 0$)

(H5) a) $I = a \cdot b$, $U = 2a + 2b = 1$, Maximiere I!

Damit $I(a) = a\left(\frac{1}{2} - a\right)$, $a \in \left(0, \frac{1}{2}\right)$

$$I'(a) = \frac{1}{2} - 2a \overset{!}{=} 0$$



$$\Rightarrow a = \frac{1}{4}$$

$$I''(a) = -2 < 0$$

$\Rightarrow$ $a = \frac{1}{4}$ ist Maximum

b) $V = h\pi r^2 \overset{!}{=} \text{const}$, $U = 2\pi r^2 + h \cdot 2\pi r$, Minimiere U!

$$\Leftrightarrow h = \frac{V}{\pi r^2}$$

Also
$$U(r) = 2\pi r^2 + \frac{2V}{r}$$

$$U'(r) = 4\pi r - \frac{2V}{r^2} \overset{!}{=} 0$$

$$\Rightarrow r = \left(\frac{V}{2\pi}\right)^{1/3}$$

$$U''(r) = 4\pi + \frac{4V}{r^3} > 0$$

$r = \left(\frac{V}{2\pi}\right)^{1/3}$ ist Maximum

$$\Rightarrow h = \frac{V}{\pi r^2} = \frac{V}{\pi} \cdot \left(\frac{2\pi}{V}\right)^{2/3} = \left(\frac{4V}{\pi}\right)^{1/3} = 2 \cdot r$$

Damit: $U = 2\pi r^2 + 2\pi r h = 6\pi r^2$

$$= 6 \frac{V^{2/3}}{(2\pi)^{2/3}}$$