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With an Eye on the Mathematical Horizon: Knowing Mathematics for Teaching to Learners' Mathematical Futures[1]

Over the last 20 years, we and our colleagues have been developing a 'practice-based theory' of the mathematical resources entailed by the work of teaching. Our aim is to understand better the mathematical demands of helping pupils learn mathematics. Our central hypothesis is that examining the work of teaching directly is a useful way to understand these mathematical demands (Ball & Bass, 2003). To do this, our analyses seek to identify recurrent tasks of teaching mathematics and to analyze their mathematical entailments. In this paper, we describe a recent development of an aspect of our theory that centers on a kind of mathematical 'peripheral vision', a view of the larger mathematical landscape, that teaching requires. We call this kind of vision *horizon knowledge* of mathematics and we consider it a part of mathematical knowledge for teaching. The following vignette illustrates how the need for horizon knowledge can arise in teaching.

> Students in a grade 1 class were measuring their handprints as part of an exploration of the notion of area. They traced the outline of their hands on graph paper, and then counted the number of square cells contained inside the hand outline. One child suggested getting the graph paper used by older pupils because the squares were much smaller and they would be able to get a closer count of the area of their handprints. The teacher, who happened to have recently studied integral calculus, heard the comment as reflecting a surprising intuitive grasp of the fundamental idea that finer mesh affords more accurate measurement. The teacher decided to call the child's idea to the attention of the class and asked what others thought. Many grasped the point and were intrigued, and several wanted to go get some of the smaller-grid paper and try her idea. When they returned with some sheets of the finer-grid graph paper, they tried using it to measure their hands and commented excitedly that they seemed to be able to count more precisely with the finer mesh paper.

Although the class did not pursue the idea in depth, this encounter with the roots of the powerful notion of limits nonetheless exemplifies the importance of teachers being able to hear their students and to build bridges between their thinking and fundamental ideas and practices of the discipline. Such knowledge afforded the teacher in the vignette a view of the mathe-

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[1] This paper is based on a keynote address at the 43rd Jahrestagung für Didaktik der Mathematik held in Oldenburg, Germany, March 1 – 4, 2009. The ideas in the paper are ones that have been influenced by the members of our research group, to whom we are grateful, including especially Mark Thames and Laurie Sleep.

matical horizon with which she was able to interpret the child's question and to make an informed decision about how and how much to take it up.

Responsibilities and dilemmas of mathematics teaching

Our notion of teaching entails three central responsibilities: (1) To provide effective opportunities to learn substantial mathematics, and treat the mathematics with intellectual integrity (Bruner, 1960); (2) to be able to hear student thinking, take it seriously, and make it an integral part of the instruction; and (3) to be committed to the learning of every student, and further to the learning of the class as an intellectual community.

These responsibilities can pull in different directions. Reconciling them presents teachers with fundamental dilemmas of teaching: How to balance mathematical rigor with generosity toward emergent student thinking, how to manage mathematical opportunities wisely, or how to make teaching both responsible and responsive? The capacity to do this work entails considerable knowledge and skill, some of it deeply content-based. What is the mathematical knowledge and skill that it takes to manage these issues in teaching, and how can one prepare teachers with opportunities to learn the content in such ways? We turn next to our approach to these questions, which builds on the notion of pedagogical content knowledge (PCK) developed by Lee Shulman and his colleagues (Shulman, 1986).

Mathematical knowledge for teaching (MKT)

'Horizon knowledge,' the focus of this paper, is a sub-domain of what we call *mathematical knowledge for teaching* (MKT), a practiced-based theory of the knowledge of mathematics entailed by the work of teaching mathematics.[2] To provide context for our discussion, we start with a synopsis of our research on MKT. Our work comprises this set of activities:

- Empirically study instruction to identify the mathematical *work* of teaching (Ball & Bass, 2003).
- Analyze what mathematical knowledge is entailed by this work to form a hypothetical characterization of MKT (Ball, Hill, & Bass, 2005; Ball, Thames, & Phelps, 2008).

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[2] This is the product of work of several research groups at the University of Michigan, over the past 15 years. It has included the development of survey measures of MKT (mainly at the K-8 grade levels), linking MKT to student learning gains, development of video-codes for the "mathematical quality of instruction" and linking this too to MKT measures, and now also the infusion of MKT ideas into curriculum for teacher education and professional development.

– Test this theory of MKT by developing measures of MKT (Hill, Schilling, & Ball, 2004), validating teacher scores against practice (Hill, Blunk, Charalambos, Lewis, Phelps, Sleep, & Ball, 2008) and against student achievement gains (Hill, Rowan, & Ball, 2005).
– Develop and evaluate approaches to helping teachers learn MKT (Ball, Sleep, Boerst, & Bass, 2009)

What is MKT and how is it distinct from 'pure' knowledge of content? How is it different from knowing ways to teach the content?

We illustrate with an example, situated in two-digit multiplication. Consider the multiplication problem 49×25. The calculation produces the answer 1225. In teaching, being able to multiply is necessary, but far from enough. Being able to get the answer 1225 equips the teacher to recognize whether a student's answer is right or wrong, but little else. When pupils make errors, something that happens regularly in teaching, diagnosis is the basis for knowing how best to intervene. What is involved in figuring out what students have done? Try analyzing the following incorrect answers:

(a)	(b)	(c)
$\begin{array}{r} 49 \\ \underline{\times\, 25} \\ 405 \\ \underline{108} \\ 1485 \end{array}$	$\begin{array}{r} 49 \\ \underline{\times\, 25} \\ 225 \\ \underline{100} \\ 325 \end{array}$	$\begin{array}{r} 49 \\ \underline{\times\, 25} \\ 1250 \\ \underline{25} \\ 1275 \end{array}$

First, it is important to note that such diagnosis is a common task of teaching. One might argue that the best procedure would be to probe pupils' thinking through individual interactions to better understand how they were reasoning about the problems. But what if students are not present when the teacher is examining the work, say, as part of marking homework papers? In the absence of knowledge of the student, is there some purely mathematical way of diagnosing these errors, based just on the numerical data? And, if so, what is the mathematical knowledge deployed in doing so?

Interestingly, such analysis is something that many skillful teachers perform with great facility, while it is difficult for other mathematically trained professionals (including mathematicians) who do not teach children. It is for this reason that we call the knowledge needed to do this '*specialized*' knowledge of mathematics; it is mathematical knowledge specialized in particular for the work of teaching.

We turn to annotate these three examples for the reader. Example (c) appears to be a compensation strategy, replacing 49 by 50, to get the (easier) product 50 x 25 = 1250, but then adding, rather than subtracting 25 to compensate for the change. Example (b) can be seen to be a conventional and widely used algorithm for multiplication, but deployed 'upside down': first 9 x 25 = 225; then 4 x 25 = 100. But this second product is really 40 x 25 = 1000, and so

> the 100 is actually 100 *tens*, which should be recorded one place over. Finally, example (a) is somewhat subtle. One guess is that the product 5 x 9 = forty five may have been recorded as 405 (40 + 5) and similarly 2 x 9 = eighteen was recorded as 108 (10 + 8). This hypothesis would imply that the solver ignored the need to multiply by the 4 in 49. A more persuasive hypothesis is that in carrying out the standard U. S. algorithm, first multiply 5 x 9 = 45, record the 5 in the units position, and carry the 4 to the tens column. The next step normally would be to multiply 5 x 4 = 20, and then add the carried 4 to obtain 24. However, here the 4 was added first to the 4 in 49, yielding 4 + 4 + 8, and then multiplied by 8 to get 40. This hypothesis can be tested by seeing what happened with the next multiplications, by 2: 2 x 9 = 18; record the 8 and carry the 1. Then add the 1 to 4 to get 5 and multiply 2 x 5 = 10. Performing this analysis requires a shift in perspective from one's own accustomed procedures, a skill of mathematical analysis centrally required in teaching.

This skill is useful not only when students make errors but also when they get right answers using unrecognized methods. Is the right answer based on a method, or a lucky accident? Does the method work in general? If not, under what conditions does it work?

Other examples of tasks of teaching that require mathematical knowledge and skill include:

- Selecting/designing instructional activities
- Identifying and working toward the mathematical goal of the lesson
- Listening to and interpreting students' responses
- Analyzing student work
- Teaching students what counts as "mathematics" and mathematical practice
- Making error a fruitful site for mathematical work
- Attending to ambiguity of specific words
- Deciding what to clarify, make more precise, leave in student's own language

The domains of MKT (The "egg")

Our analyses of the work of teaching, combined with our empirical analyses of teachers' knowledge and reasoning in the context of the work of teaching have produced a framework that articulates distinct 'domains' of MKT (Ball, Thames, & Phelps, 2008) – see Fig. 1.

Our notion of MKT comprises two of the categories of knowledge defined by Shulman and his colleagues: pedagogical content knowledge (PCK); and content knowledge (CK) (Shulman, 1986). In our work, we have refined the earlier characterizations, particularly on the side of content knowledge. Following Shulman, we define PCK as a blend of knowledge of content and knowledge of pedagogy. Inside this we distinguish knowledge of content and students (for example typical student errors), knowledge of content and teaching (for example with what sequence of examples to introduce a new concept or method), and knowledge of content and curriculum (for example educational goals, standards, state assessments, grade levels where particular topics are typically taught, etc.)

On the content knowledge side, perhaps our most significant work has been the identification and measurement of specialized content knowledge, discussed above. This is complementary to what we call 'common content knowledge.' By 'common,' we mean knowledge held *in common* with professionals in other mathematically intensive fields.

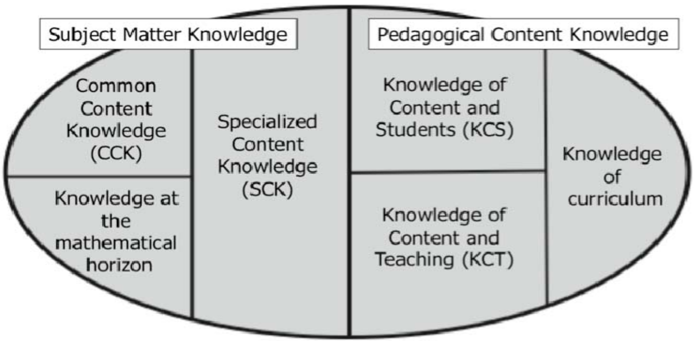

Figure 1

But we have known from the beginning that there is a kind of content knowledge that is neither common nor specialized. It is not directly deployed in present instruction, yet it supports a kind of awareness, sensibility, disposition that informs, orients and culturally frames instructional practice. We consider this a kind of peripheral vision, or awareness of the mathematical horizon (Ball, 1993). We turn next to discuss our emerging concept of 'horizon knowledge.'

Horizon knowledge of mathematics

Widely-known is Bruner's (1960) assertion that it is possible to teach any subject to any learner in an intellectually honest way—a claim that inspires curriculum developers and teachers to consider ways to engage pupils in the seeds of big and complex ideas. Schwab (1961/1978) too, argued for the importance of acquainting pupils with the major structures of a discipline as a foundation for their appreciation of its major ideas and ways of knowing. Our interest in the relation of the discipline to the teaching and learning of any particular age is in sympathy with the ideas of Bruner and Schwab, but arises directly from our studies of practice. Repeatedly we see that connections emerge and ideas touch, even across great expanses of regular curricular sequence. We see that teaching requires a sense of how the mathematics at play now is related to larger mathematical ideas, structures, and principles. Some may be ones that students will learn in later grades; some may be at the heart of what doing mathematics is. Attention to the mathematical horizon is thus important in orienting instruction to embody both pedagogical foresight and mathematical integrity. Having this sort of knowledge of the mathematical horizon can help in making decisions about how, for example, to draw the number line or how to set the anticipation that the number line will soon 'fill in' with more and more numbers. This sort of knowledge can help teachers think about how to anticipate the integers when their pupils know only whole numbers, or to set the cognitive stage for real numbers. But this kind of vision is not the same as the detailed curricular knowledge we include in PCK (knowledge of mathematics and curriculum). We define horizon knowledge as an awareness – more as an experienced and appreciative tourist than as a tour guide – of the large mathematical landscape in which the present experience and instruction is situated. It engages those aspects of the mathematics that, while perhaps not contained in the curriculum, are nonetheless useful to pupils' present learning, that illuminate and confer a comprehensible sense of the larger significance of what may be only partially revealed in the mathematics of the moment. It is a kind of knowledge that can guide the following kinds of teaching responsibilities and acts:

- Making judgments about mathematical importance
- Hearing mathematical significance in what students are saying
- Highlighting and underscoring key points
- Anticipating and making connections
- Noticing and evaluating mathematical opportunities
- Catching mathematical distortions or possible precursors to later mathematical confusion or misrepresentation

Our current conception of horizon knowledge has four constituent elements:

1) A sense of the mathematical environment surrounding the current "location" in instruction.
2) Major disciplinary ideas and structures
3) Key mathematical practices
4) Core mathematical values and sensibilities

We turn next to examine an episode of teaching, and identify some kinds of horizon knowledge that can be seen in what transpires.

A teaching episode: ‘Sean Numbers’[3]

The setting is a grade 3 classroom, comprising culturally and linguistically diverse pupils, many speaking English as a second language. The children have been working on even and odd numbers. Before third grade they had already learned that certain small numbers were called ‘even’ and others ‘odd,’ but they lacked a formal definition of these concepts. One day, one of the boys, Sean, raises his hand and says, ‘I was just thinking about six. … I’m just thinking it can be an odd number, too, ’cause there could be two, four, six, and two, three twos, that’d make six. … And two threes, that it could be an odd and an even number. Both! Three things to make it, and there could be two things to make it.’ The reader may form ready opinions about Sean’s thinking or wish to assert what the teacher should do. (‘Why doesn’t the teacher just point out his error and explain it?’) But, before even considering such judgments, consider the more difficult questions, ‘What is significant mathematically about what is going on in this episode? What might be helpful for a teacher to see, and be sensitive to, mathematically?’

The teacher does not immediately challenge or correct Sean. She re-voices and tries to publicly clarify what he is saying, and then invites comments from the class. His classmates quickly disagree. They already know from second grade that 6 is even. We follow now the mathematical debate as it unfolds in the class, and attend to how the children are processing mathematical ideas and claims, and to the mathematical moves of the teacher to orchestrate their discussion. Cassandra, the first to object, points to the number line above the blackboard, saying, ‘Six can’t be an odd number because this is (she points to the number line, starting with zero) even, odd, even, odd, even, odd, even, ...’ ’Because zero’s not an odd number.’

Sean persists, ‘Because there can be three of something to make six, and three of something is odd.’ Kevin protests, ‘That doesn’t necessarily mean that six is odd. ... Just because two odd numbers add up to an even number doesn’t mean it has to be odd.’ The teacher asks, ‘What’s our working definition of an even number? Jillian explains, ‘If you have a number that you can split up evenly without having to make (long pause) to split one in half, then, it’s an even number.’ When the teacher then asks Sean if he can do that with six, he agrees, and she says, ‘So then it would fit our working definition; then it would be even, okay?’ Sean nods but adds, ‘And it could be odd. Three twos could make it.’

Sean, contrary to the tacit understanding of the class, seems to allow that a number can be both even and odd. The teacher realizes that this discussion requires an explicit definition of odd numbers. After some discussion, the class agrees that odd numbers are those you cannot split up fairly into two groups. But Sean is tenacious. ‘You could split six fairly (two threes) and not fairly (three twos).’

[3] See Ball (1993) for an analysis of this episode as a case of teaching ‘with an eye on the mathematical horizon’.

Mei, listening carefully, exclaims, ‘I think I know what he is saying! I think what he’s saying is that you have three groups of two. And three is a odd number so six can be an odd number and an even number.’ Sean nods — this is what he is saying. The teacher asks if others agree. Mei raises her hand. ‘I disagree with that because it’s not according to like … here, can I show it on the board?’ She continues, ’It’s not according to like … how many groups it is. Let’s say that I have (long pause while she thinks) let’s say – ten.’ She draws on the board. ‘Here are ten circles. And then you would split them … by twos. … One, two, three, four, five …’ (Mei draws lines between each pair of circles and counts the groups of two.) ‘Then why do you not call ten a, like … an odd number and an even number, or why don’t you call other numbers an odd number and an even number?’ To Mei’s surprise, and then dismay, Sean responds, ‘I disagree with myself. I didn’t think of it that way. Thank you for bringing it up; so, I say it’s … ten can be an odd and an even.’ Many pupils raise their hands, eager to protest. Mei, intending to prove to Sean that his reasoning fails when the claim is extended to other numbers, has instead succeeded in giving Sean an expanded understanding and appreciation of his own idea, which he embraces gratefully. Mei protests, ‘But what about other numbers?! If you keep on going on like that and you say that other numbers are odd and even, maybe we’ll end it up with all numbers are odd and even. Then it won’t make sense that all numbers should be odd and even, because if all numbers were odd and even, we wouldn’t even be having this discussion!’

Mathematical knowledge at the horizon of this lesson

In this section we provide commentary on the aspects of horizon knowledge that may relate to the episode. A synopsis, keyed to the four elements, is represented in this table:

Element	Possible examples from this episode
1) A sense of the mathematical environment surrounding the current “location” in instruction.	Definitions, factorization, modular arithmetic
2) Major disciplinary ideas and structures	Number systems, even and odd, powers, number theoretic concepts
3) Key mathematical practices	Establishing correspondences and equivalence, choosing representations, questioning, using definitions, proving
4) Core mathematical values and sensibilities	Precision, care with mathematical language consistency, parsimony, coherence, connections

First, worth noting is that the episode is not only about even and odd numbers, but also centrally about mathematical communication, reasoning and proving (3). The children are making and defending and critiquing mathematical claims. The critiques are based on distinct definitions of even and odd. In fact, though the class has not developed formal definitions, there are three types of mathematical definitions of even and odd numbers that are implicitly in play in the class discourse (4). For example, for evenness:

Fair-share: A number is even if it can be made into to two equal groups with none left over.

Pair: A number is even if it can be made into groups of two with none left over.

Alternating: Starting with 0, the numbers alternate, even, odd, even, … .

For odd numbers, one has one left over in the fair-share and pair definitions. Note that the alternating definition immediately implies that a number cannot be both odd and even. These definitions are not obviously equivalent. To reconcile arguments resting on distinct definitions would first require reconciliation of the definitions themselves (3, 4). For example the equivalence of fair-share and pair is a special case of the commutativity of multiplication, which could be enacted using rectangular arrays (2).

Mei gives a clear articulation of Sean's reasoning, even though she disagrees with it. More than challenging his method ('It's not according to like how many groups of two there are'), she shows how his own reasoning would lead to conclusions that she expects would be unacceptable even to Sean, a form of reasoning by contradiction. To do this she generalizes (3, 4) Sean's idea about 6 to the class of all numbers that, like six, are made of an odd number of groups of two. Mei's sensibility about definitions – that they fail in their purpose if they lose the capacity to make significant distinctions, to give concepts appropriately sharp boundaries (4) – convinces her that there are many even-and-odd numbers that Sean's idea would usher in. She expects this to unsettle him. But Sean both understands and welcomes the possibility.

What about Sean's thinking? Though Sean misuses the terms 'even' and 'odd,' he nonetheless has a clear and meaningful mathematical idea about six: it has 'an odd way of being even.' But, lacking vocabulary to name this feature, he appropriates the name 'odd-and-even' for it. While Sean is thinking only about six, Mei generalizes this to odd multiples of two. What are these numbers (that the teacher later names 'Sean numbers' (3)) introduced by Mei? Even and odd are about mod 2 modular arithmetic. Sean and Mei have opened a glimpse of mod 4 arithmetic, identifying numbers that are congruent to 2 mod 4 (2). These numbers were already studied by the ancient Greeks, and turn out also to be exactly those natural numbers that are not a difference of two squares[4]. So Sean's idea has some interesting mathematical significance that he could not have anticipated, but that might a teacher might notice. Indeed, once Mei had identified these numbers, the students began an exploration of their properties, identifying their patterns (they

[4] The squares mod 4 are 0 and 1, so the differences of two squares mod 4 are 0, 1, and -1 (or 3), but not 2. If an integer N is ± 1 mod 4, then it is odd, so $N = 2n+1 = (n+1)^2 - n^2$. If N is 0 mod 4, then $N = 4n = (n+1)^2 - (n-1)^2$.

appear every fourth number, starting with 2); making and proving conjectures (a sum of Sean numbers does not produce a Sean number); etc. The children are practicing skills of mathematical exploration, reasoning, and generalization (3).

From this example, we see that even a simple ten-minute segment in a grade 3 class can take pupils and their teacher into contact with some major mathematical ideas and practices. Noticing the significance of the discussion would alter a teacher's view of the pupils' ideas and, although it might not compel her to spend time following them, it could nonetheless shape her view of the value of time spent unpacking and pursuing what first seemed like a simple pupil misconception. Seeing that students may say things that sound wrong, but that on closer listening are insightful is one use of horizon knowledge. Another is in pausing to develop pupils' sensibilities with mathematical habits of mind ('if you keep on going on like that' can be heard as a child's version of the mathematically crucial impulse to generalize) or to raise questions that are fundamental to the mathematical enterprise ('What is our working definition?'). It is horizon knowledge that can orient the teacher to hear, to speak, and to make decisions that honor children's often surprisingly deep insights that anticipate their later mathematical journeys. The broader mathematical integrity that horizon knowledge can enable in teaching does not imply that teachers will rush to that horizon with their pupils. Indeed to do so would often be both pedagogically reckless and irresponsible. But our explorations and preliminary analyses suggest that horizon knowledge can open up for teachers a greater appreciation both of their pupils' insights and of the significance of their own mathematical settings.

Current thoughts about horizon knowledge as a domain of MKT

Felix Klein (1924) offered the attractive and oft-cited idea of 'a higher perspective on elementary mathematics.'[5] Our notion of 'horizon knowledge' complements his. We hypothesize it as a kind of elementary perspective on advanced knowledge that equips teachers with a broader and also more particular vision and orientation for their work. This may include the capacity to see 'backwards,' to how earlier encounters inform more complex ones, as well as how current ones will shape and interact with later ones. It is also about sheer mathematical honesty—that sense of the territory that helps to bring a sense of judgment and good taste to teachers' responsibilities toward their pupils. Some horizon knowledge is about topics, some is about practices, and some is about values. Important, too, is that it is not knowledge of the kind that teachers need to understand in order to explain it to pupils; similarly, knowledge of the horizon does not create an imperative to act in any particular mathematical direction.

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[5] More often translated from the German 'höhere' as 'advanced,' we appreciate Jeremy Kilpatrick's (2008) interpretation of the word as signifying higher, as in a 'higher perspective,' or 'eagle's eye view' from the discipline of the school curriculum.

There are many important issues that are as yet undeveloped or unresolved and form the directions in which we and our colleagues are now working. As appealing as the notion may be, for example, we have no evidence that such mathematical perspective produces improvements in teachers' effectiveness or in pupils' learning. We do not know how to estimate how far out or in what direction the pedagogically relevant and useful horizon extends. We do not know the level of detail that is needed for horizon knowledge to be useful. Moreover, we do not know how horizon knowledge can be helpfully acquired and developed, and we do not, as yet, have ways to assess or measure it.

Much remains to be done. We think, however, that teaching can be more skillful when teachers have mathematical perspective on what lies in all directions, behind as well as ahead, for their pupils, that can serve to orient their navigation of the territory. We seek to work toward conceptualizing more precisely what comprises that sense of horizon

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